12.815 Problem Set 1
Radiative Transfer Background
Due Wednesday September 23, 2015

1. Radiation terminology (Liou Chapter 1)

Define the terms and units of the following, and note which terms are interchangeable. Organize your answer however you see fit.
Intensity, monochromatic intensity, radiance, flux density, irradiance, radiant power, emittance, brightness, luminance, luminosity, total flux.

2. Intensity and Flux

a. (Liou 1.1) What is the meaning of isotropic radiation? Show that for isotropic radiation the monochromatic flux integrated over a hemispheric solid angle is
$F_\lambda = \pi I_\lambda$
Use a spherical polar coordinate system as in Liou Figure 1.3.

b. Show that for isotropic radiation the flux integrated over the entire solid angle is $F_\lambda = 0$, and that $I_\lambda = J_\lambda$.

c. Given that the intensity I_λ does not depend on distance (in a medium with no extinction or emission), show that the flux obeys the inverse square law, $F_\lambda \sim 1/d^2$ where d is distance from the source.

3. Blackbody Radiation

a. Start with the expression for Planck's law per unit frequency and convert to Planck's law per unit wavelength.

b. Derive Wien's Law

$$\lambda_{\max} = \frac{b}{T}$$

$$b = 2.90 \times 10^{-3}\,\mathrm{mK}$$

via $\dfrac{dB(\lambda,T)}{d\lambda} = 0$.

c. Derive the Stefan-Boltzmann constant, σ in the below equation, in terms of fundamental constants (Planck's constant h, Boltzmann's constant k, and the speed of light c), by integrating the Planck function over the entire wavelength domain,

$$\int_0^\infty B_\lambda d\lambda = \frac{\sigma T^4}{\pi}$$

Use the value of the integral $\int_0^\infty \frac{x^3}{\exp(x)-1} = \frac{\pi^4}{15}$.

d. (Liou Question 1.6) An infrared scanning radiometer aboard a meteorological satellite measures the outgoing radiation emitted from Earth's surface in the 10 micron window region. Assuming that the effect of the atmosphere between the satellite and the surface can be neglected, what is the surface temperature if the observed radiance at 10 microns is 9.8 W m^{-2} μm^{-1} sr^{-1}?

e. Is it realistic to assume that the effect of the atmosphere between the satellite and the surface can be neglected in part d?

4. Local Thermodynamic Equilibrium

Give a description of LTE, in one or two paragraphs. Include the following: What is LTE? Under what conditions is LTE valid? At which altitude in Earth's atmosphere does LTE become invalid? Why?

5. Hydrostatic Equilibrium

a. Derive the hydrostatic equilibrium equation,

$$\frac{dP}{dz} = -g\rho$$

by considering the pressure forces acting on a volume element balanced by gravity.

b. Derive the pressure scale height *H*, for constant temperature by starting from the equation in part a to show

$$P = P_0 e^{-z/H}$$

where P_0 is the reference pressure (pressure at sea level) and z is altitude.

c. What is the scale height near Earth's surface? How much lower is the air density on top of Mt. Everest compared to sea level?

d. What is the scale height for a super Earth exoplanet (defined in this example as a 3 Earth mass, 1.75 Earth radius planet), with an atmospheric temperature similar to Earth's. but a composition dominated by molecular hydrogen (H_2)? Would Mt. Everest climbers have any problem running out of air?

12.815 Problem Set 2
Radiative Transfer and Thermal Emission Spectra
Due Wednesday September 30, 2015

<u>1. Does reflected solar flux have a significant contribution to Earth's thermal emission? (Problem 6.14 in Petty)</u>

Calculate the upward flux of reflected solar radiation at 12 microns, assuming an overhead sun (temperature 6000 K, solid angle $\Delta\omega$ = 6.8e-5 sr) and a surface (flux) reflectivity of 10%. Find the ratio of the above flux to that due to terrestrial emission at a temperature of 300 K.

For problems 2 and 3, use the David Archer MODTRAN Interface (same as used in class) at http://climatemodels.uchicago.edu/modtran/

<u>2. Earth Atmospheric Emission to Space</u>

<u>2.1 Clear atmosphere emission spectrum</u>

Use the default parameters to compute the vertical, outgoing flux in W/m^2 at the top of the atmosphere in the wavenumber range provided. The default setting should be no clouds or rain. Save or print out the plot. Identify as many absorbing and emitting features as you can. Be sure to note if the spectral features are absorption or emission features.

<u>2.2 Cloudy emission spectrum</u>

Do clouds change the spectral features? If so, how?

<u>3. Earth Atmospheric Emission</u>

<u>3.1 Earth's atmosphere emission looking down.</u>

Plot Earth's atmosphere emission looking down from different altitudes. Start with 0 km and then increase by 2 km up to 10 km then by 10 km up to 40 km then at 50 km and 70 km.

Describe how features and the continuum change and why. Submit a few of the plots to support your description (no need to include all).

<u>3.2 Earth's atmosphere emission looking up from different altitudes.</u> Same as above but have the atmosphere sensor looking up.

<u>3.3 How are your results interpreted using the thermal emission equation derived in class?</u>

$$I(0) = I(\tau')e^{-\tau'} + \int_0^{\tau'} B(\tau)e^{-\tau}\,d\tau$$

4. Surface Temperature

Consider an atmosphere that is completely transparent to shortwave (solar) radiation, but very opaque to infrared (IR) terrestrial radiation.

a. Consider the 2-slab simple greenhouse model atmosphere discussed in class. Show that the surface temperature (T_s) and the emission temperature (T_e) are related by

$$T_s = 2^{1/4} T_e$$

b. We now assume that the atmosphere is a grey body and has a constant emissivity averaged over all wavelengths ε, i.e. the atmosphere only absorbs/emits a proportion ε of what a blackbody atmosphere would absorb/emit. Show that the surface temperature (T_s) and the emission temperature (T_e) are related by

$$T_s = \left(\frac{2}{2-\varepsilon} \right)^{1/4} T_e$$

c. For the leaky greenhouse, derive a relationship between the atmosphere temperature T_a, and T_s, and T_e.

d. Which layer is always the hottest?

Now assume that the atmosphere can be represented by *N* slabs of atmosphere, each of which is completely absorbing of IR, as depicted in Figure 2.

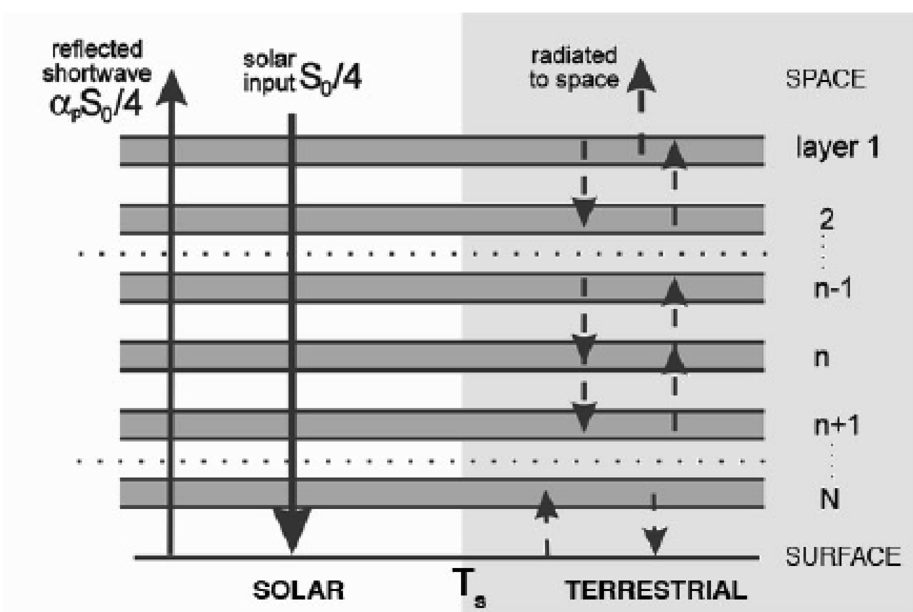

Figure 2. An atmosphere made up of N slabs each of which is completely absorbing in the IR. (From Plumb and Marshall; not all layers are shown.)

e. By considering the radiative equilibrium of the surface, show that the surface must be warmer than the lowest atmospheric layer.

f. By considering the radiative equilibrium of the n^{th} layer, show that, in equilibrium,

$$2T_n^4 = T_{n+1}^4 + T_{n-1}^4, \qquad (1)$$

where T_n is the temperature of the n^{th} layer, for $n>1$. Hence argue that the equilibrium surface temperature is

$$T_s = (N+1)^{1/4} T_e$$

where T_e is the planetary emission temperature. [Hint: Use your answer to part (e); determine T_1 and use equation (1) to get a relationship for temperature differences between adjacent layers.]

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Atmospheric Radiation and Convection

Problem Set 3

1. A two-dimensional line continuously supplies buoyancy to an unstratified, Boussinesq fluid at a rate given by F . By drawing an analog to the maintained point source of buoyancy discussed in class, determine the dimensions of F in terms of the primary quantities length and time. Now, using dimensional analysis, find the dependencies of the plume's radius, buoyancy, and vertical velocity on F and on the distance z above the line source. Assume that the plume is fully turbulent.

2. Estimate the maximum altitude to which the thermal arising from a 1-megaton bomb explosion near the surface will rise through a calm, Boussinesq atmosphere. Assume that all the energy from the explosion is turned into heat and ignore exotic effects such as breakdown of the Boussinesq and continuum approximations, magneto-hydrodynamical effects, and continued heating from, e.g., radioactivity and condensation. Assume the following:

   a. Buoyancy frequency of the stratified atmosphere is $10^{-2} s^{-1}$

   b. Entrainment parameter $\alpha = 0.285$

   c. 1 megaton (of dynamite) $= 4 \times 10^{15}$ $Joules$

12.815 Problem Set 4
Molecular Spectroscopy
Due Tuesday October 14, 2015

1. Molecular Spectroscopy Terminology

Define the following, including units where appropriate.
Wavenumber, fundamental mode, degeneracy, overtone bands, combination bands.

2. Greenhouse Gases

a. Three gases make up 99.96% of dry air by volume: N_2 (78.1%), O_2, (20.9%) and Ar (0.93%). The major greenhouse gases are present in much smaller amounts: H_2O (less than 4%), CO_2 (385 ppm), CH_4 (1.7 ppm), and N_2O (320 ppb). Explain why despite their very low abundances, these greenhouse gases are such strong absorbers.

b. Imagine you are trying to create a designer coolant that has no negative effects on the atmosphere (e.g., CFCs are not only ozone-depleting gases but are also greenhouse gases). In order not to contribute to warming of Earth, what properties in a molecule would you aim for? What properties would you try to avoid? One or two paragraphs is sufficient to describe your answer.

c. Briefly summarize the main spectral characteristics of the major greenhouse gases in Earth's atmosphere. See Liou section 4.2 and feel free to consult any other references.

3. Molecular Rotational and Vibrational Frequencies.

a. Show that the rotational frequencies of a linear diatomic molecule are at microwave wavelengths.

b. Show that the vibrational frequencies of a linear diatomic molecule are at infrared wavelengths.

4. Molecular Energy Levels, Internuclear Separations, and Bond Constants

Figure 1 is a synthetic spectrum of the CO molecule. When possible, use equations to answer the following questions. The top two panels show the same wavelength range but a different y-axis scale. The third and fourth panels from the top show a zoom in of each of the features shown in the top two plots. In answering the below questions, feel free to annotate the graph.

a. Show that at room temperature, almost all molecules are in the ground vibrational state. What happens at the high temperatures of hot super Earths (1000 to 2000 K)?

For the CO graph:

b. Which CO vibrational transitions are shown?

c. Explain what you are seeing. Which parts are rotational and which vibrational transitions?

d. Describe the shape of the features.

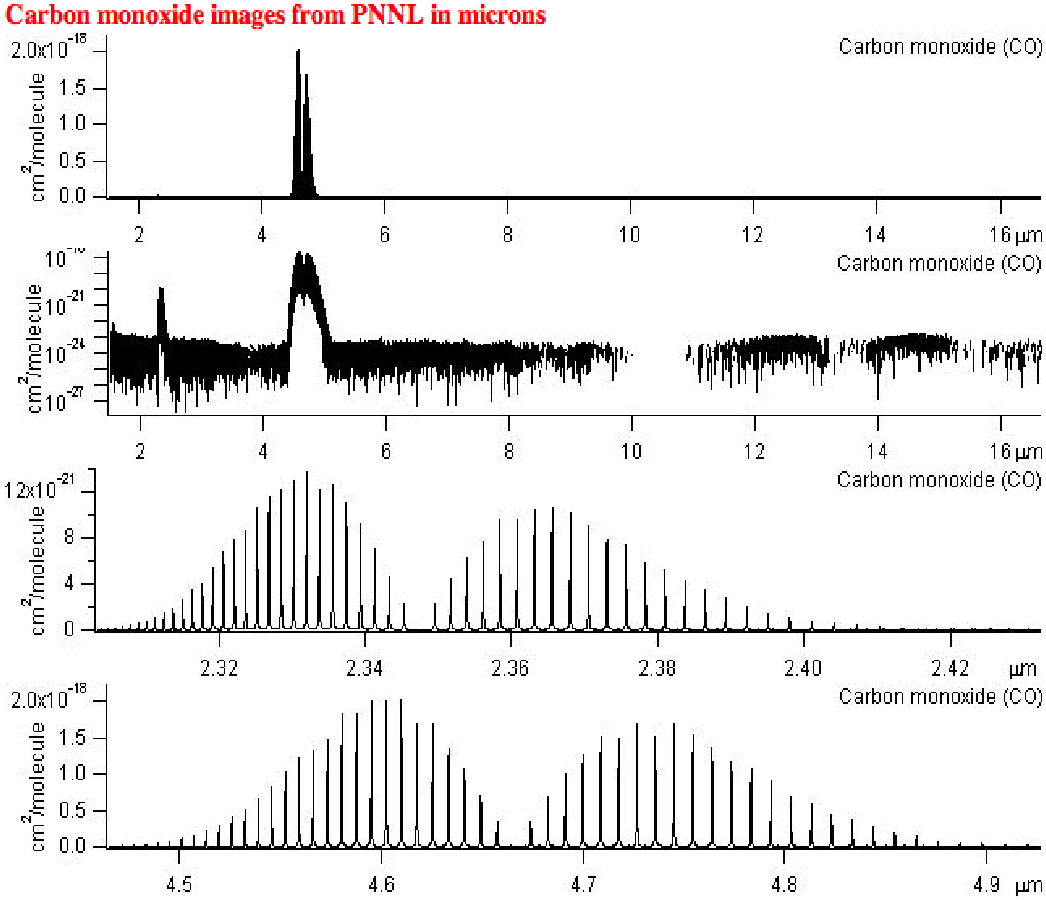
e. Describe and explain the spacing of the rotational lines.

f. Estimate the internuclear separation of CO by estimating the spacing between the rotational lines and using the masses of the nuclei.

g. Use the equation for vibrational energy states to show that transitions occur at infrared wavelengths. For CO, take the force constant as $C = 1869$ N m^{-2}.

h. In class we saw that vibrational energy levels are equally spaced and hence the transition energies are constant. Yet, the two vibrational transitions shown in the figure are at different wavelengths. Explain.

Carbon monoxide images from PNNL in microns



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Atmospheric Radiation and Convection

Problem Set 5

1. The cabins of aircraft must be continuously ventilated for the health of passengers. Suppose a jet is flying at 200 hPa where the ambient temperature is $-50^o C$. The ambient air is brought into the cabin where the pressure is maintained at 850 hPa. How much heat energy must be supplied to each kilogram of air to bring it to a temperature of $+20^o C$? Take the gas constant of the air to be $R_d = 287\ J\ K^{-1} Kg^{-1}$ and its heat capacity at constant pressure to be $c_p = 1005\ J\ K^{-1} Kg^{-1}$.

2. The breath we exhale is nearly at body temperature ($36^o C$) and has a relative humidity of about 80%. It gradually mixes with ambient air. What is the maximum temperature of the ambient air that would result in some condensation of water vapor assuming that the ambient relative humidity is a) 90% or b) 10%? Assume that the ambient pressure is 1000 hPa and use equation 4.4.14 in *"Atmospheric Convection"* (available on the course Stellar site) for the saturation vapor pressure. The ratio of the molecular weight of water to that of the other constituents of air is 0.622. You will need a calculator or computational software such as MATLAB to do this.

3. In middle latitudes, air flows generally from west to east. Suppose air just to the west of California has a temperature of $+20^o C$, a pressure of 1000 hPa, and is saturated with water vapor. It ascends the western slopes of the Sierra Nevada range to a minimum pressure of 600 hPa. Assume that all the condensed water falls out as rain, ignore the ice phase, and assume that the ascent is moist adiabatic. Then the air descends along the eastern slopes of the Sierra to a pressure of 950 hPa. Assume the descent is dry adiabatic. What will be the temperature of the air after it descends? Use the gas constants given in Problem 1 above and take $L_v = 2.5 \times 10^6\ J\ Kg^{-1}$ and the heat capacity of liquid water $c_l = 4190\ J\ K^{-1} Kg^{-1}$.

12.815 Problem Set 6: Radiative Transfer

October 20, 2015

Pr Sara SEAGER

The following problems are from *Liou, K. N. (2002) An Introduction to Atmospheric Radiation*.

1 Asymmetry factor of simple phase functions

Show that for the isotropic and the Rayleigh scattering cases, the asymmetry factor is zero.

2 Diffusion approximation: The two-stream model

We define:

- The upwards and downwards flux densities:

$$\mathcal{F}^{\uparrow\downarrow}(\tau) \stackrel{\text{def}}{=} \int_0^{2\pi} \int_0^{\pm 1} I(\tau,\mu,\phi)\mu d\mu d\phi \tag{1}$$

  where the notations $(\uparrow,\downarrow)$ respectively corresponds to $(+,-)$. These flux densities are governed by Schuster's equations:

$$\frac{d\mathcal{F}^{\uparrow}}{d\tau} = \gamma_1 \mathcal{F}^{\uparrow} - \gamma_2 \mathcal{F}^{\downarrow} - \gamma_3 \tilde{\omega} \mathcal{F}_{\odot} \exp(-\frac{\tau}{\mu_0}) \tag{2}$$

$$\frac{d\mathcal{F}^{\downarrow}}{d\tau} = \gamma_2 \mathcal{F}^{\uparrow} - \gamma_1 \mathcal{F}^{\downarrow} + (1-\gamma_3) \tilde{\omega} \mathcal{F}_{\odot} \exp(-\frac{\tau}{\mu_0}) \tag{3}$$

  where $(\gamma_1, \gamma_2, \gamma_3)$ are appropriate weighting coefficients related to multiple scattering event, $\mathcal{F}_{\odot}$ is the solar flux and $\tilde{\omega}$ is the single scattering albedo.

- The sum and the difference of the upwards and downwards flux densities:

$$\mathcal{F}_{\text{sum}} \stackrel{\text{def}}{=} \mathcal{F}^{\downarrow} + \mathcal{F}^{\uparrow} \tag{4}$$

$$\mathcal{F}_{\text{dif}} \stackrel{\text{def}}{=} \mathcal{F}^{\downarrow} - \mathcal{F}^{\uparrow} \tag{5}$$

2.1 The flux integral

Consider the case of pure scattering, referred to as *conservative scattering*, such that the single scattering albedo is one: $\tilde{\omega} = 1$, and the two first weighting coefficients are equal: $\gamma_1 = \gamma_2 = \gamma$. Define the net flux associated with the diffusive beam as follows:

$$\mathcal{F}(\tau) \stackrel{\text{def}}{=} -\int_0^{2\pi} \int_{-1}^{1} I(\tau,\mu,\phi)\mu d\mu d\phi \tag{6}$$

Show that:

$$\frac{d\mathcal{F}}{d\tau} = \mathcal{F}_{\odot} \exp(-\frac{\tau}{\mu_0}) \tag{7}$$

and that:

$$\boxed{\mathcal{F}(\tau) + \mu_0 F_{\odot} \exp(-\frac{\tau}{\mu_0}) = \text{Constant}} \tag{8}$$

This is the so-called *flux integral*: In a pure scattering atmosphere, the total flux (direct plus diffusive solar beam) is conserved.

2.2 Solution of the two-stream model for free boundary conditions

1)

Coming back to the general case $\tilde{\omega} \in \mathbb{R}_+$, show that the flux's sum and difference verify:

$$\frac{d^2\mathcal{F}_{\text{sum}}}{d\tau^2} = k^2\mathcal{F}_{\text{sum}} + Z_1 \exp(-\frac{\tau}{\mu_0}) \tag{9}$$

$$\frac{d^2\mathcal{F}_{\text{dif}}}{d\tau^2} = k^2\mathcal{F}_{\text{dif}} + Z_2 \exp(-\frac{\tau}{\mu_0}) \tag{10}$$

where we have defined:

$$k^2 \stackrel{\text{def}}{=} \gamma_1^2 - \gamma_2^2$$

$$Z_1 \stackrel{\text{def}}{=} -[\gamma_1 + \gamma_2 + \frac{1 - 2\gamma_3}{\mu_0}]\tilde{\omega}\mathcal{F}_\odot$$

$$Z_2 \stackrel{\text{def}}{=} -[(\gamma_1 - \gamma_2)(1 - 2\gamma_3) + \frac{1}{\mu_0}]\tilde{\omega}\mathcal{F}_\odot$$

2)

Show that the general form of the solutions to equations 9 and 10 is given by:

$$\mathcal{F}^\uparrow = Kv\exp(k\tau) + Hu\exp(-k\tau) + \alpha\exp(-\frac{\tau}{\mu_0}) \tag{11}$$

$$\mathcal{F}^\downarrow = Ku\exp(k\tau) + Hv\exp(-k\tau) + \beta\exp(-\frac{\tau}{\mu_0}) \tag{12}$$

where we have defined:

$$v \stackrel{\text{def}}{=} \frac{k + \gamma_1 - \gamma_2}{2k} \tag{13}$$

$$u \stackrel{\text{def}}{=} \frac{k - (\gamma_1 - \gamma_2)}{2k} \tag{14}$$

$$\alpha \stackrel{\text{def}}{=} \frac{\mu_0^2(Z_1 + Z_2)}{2(1 - \mu_0^2 k^2)} \tag{15}$$

$$\beta \stackrel{\text{def}}{=} \frac{\mu_0^2(Z_1 - Z_2)}{2(1 - \mu_0^2 k^2)} \tag{16}$$

What are K and H?

3)

Assuming no diffusive components from the top ($\tau = 0$) and the bottom ($\tau = \tau_1$) of the atmosphere, find the solutions ($\mathcal{F}^\uparrow, \mathcal{F}^\downarrow$).

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Atmospheric Radiation and Convection

Problem Set 7

(Note: Please re-download and unzip the sounding plot package from the course Stellar site as there have been some modifications to the routines. Also note that you will need an internet connection to do this assignment.)

1. Run the MATLAB script *entrain* using station identifier 72357 (Norman, Oklahoma, USA) for the date [2011 5 10 0]. Lift parcels from 1000 hPa.
    a. Qualitatively describe the stability of the atmosphere represented by this profile to upward displacements of air from near the surface.
    b. By entering ‘0’ followed by <return> at the MATLAB prompt you will see vertical profiles of parcel buoyancy for reversible, pseudo-adiabatic and entraining plume displacements, and a table of CAPE values associated with these processes will be listed in the command window. Describe how these three buoyancy profiles and CAPE values differ from one another and why.
    c. State whether you think deep convection was present when this sounding was taken, explaining your reasoning.
2. Repeat problem 1 but use station number 06011 (Tórshavn, Faroe Islands, north of Scotland) at time [2015 1 10 12], lifting a test sample from 975 hPa.
3. Repeat problem 1 using station 91376 (Majuro, in the Marshall Islands of the tropical western North Pacific) at time [2015 10 27 0], lifting a test ample from 1000 hPa.
4. Compare and contrast the soundings from the three locations in problems 1-3. What geographical factors (location, season) do you think makes them different? It might help to locate the three places on a map if you are not already familiar with their locations.

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Atmospheric Radiation and Convection

Problem Set 8: Radiative-Convective Equilibrium

1. Using the MIT single-column model, available on http://rcmodel.mit.edu/advanced.html, perform and interpret the following experiments:

a. While leaving the other parameters at their default values, double, triple and quadruple the concentration of CO_2 from its default value and compare the results among these experiments as well as the control. What happens to surface temperature, surface precipitation, and surface shortwave and longwave fluxes (see table) and why?

b. While leaving the other parameters at their default values, double the ozone concentration. What happens to the surface temperature and to the profile of temperature through the whole atmosphere, and why?

c. While leaving the other parameters at their default values, use diurnally varying radiation" and run the model to equilibrium. Compare the 25-day average quantities to the control run. What happened to mean temperature and precipitation? Now, having checked off the box labelled "Restart from end of previous simulation" run the model for another 5 days and examine the graphical output. Interpret the diurnal behavior of temperature and precipitation, paying particular attention to the phase of the diurnal cycles of each.

d. Repeat *c* above but change the fraction of surface covered by water to 0.1 In this case, also examine the equilibrium profiles of temperature and relative humidity and interpret their differences from the experiment performed in *c* above.