12.801 Large Scale Ocean Dynamics

Problem set 1: due date: March 1, 2016

1. Consider an ocean of uniform density subject to a time dependent surface wind stress of the form $(\tau_x, \tau_y) = A\cos(\omega(t-t_0))$, where A is the amplitude of the wind stress, ω the frequency of the wind fluctuations, and t_0 is some initial time. Solve for the Ekman spiral that develops in this problem.

2. Consider an ocean with a surface mixed layer of constant buoyancy b_1 and depth h overlying an abyss of constant buoyancy b_2 and depth H. The ocean is horizontally homogeneous. Discuss the temporal evolution of the mixed layer buoyancy and depth, if the mixed layer is subject to a buoyancy loss $\mathcal{B}$.

3. Consider a two dimensional eddy field given by the streamfunction,

$$\psi = \Psi \sin(kx)\,\sin(ky), \qquad \boldsymbol{u} = (u, v) = (-\psi_y, \psi_x), \tag{1}$$

i.e. a periodic array of eddies of size $l \times l$ ($k = 2\pi/l$). You can think of this pattern as a simple representation of convecting cells in the mixed layer. You will be asked to compute the eddy diffusivity for a tracer stirred by this eddy field. You can obtain the eddy diffusivity using a simple trick. Suppose that a large scale uniform gradient $\boldsymbol{G}$ is externally imposed on the tracer, you can then proceed to calculate the tracer eddy flux $\boldsymbol{F}$ which is associated with $\boldsymbol{G}$. In terms of the notation used in class, the linear gradient represents the large scale tracer gradient, the eddy field represents the turbulent velocity field, and there is no mean velocity field, *i.e.* $\langle \boldsymbol{u} \rangle = 0$.
The procedure of assuming a steady large scale gradient is used to bypass the initial value problem and deal with a simpler steady state problem,

$$J(\psi, c) = \kappa \nabla^2 c. \tag{2}$$

The first step is to make the simple substitution,

$$c = \boldsymbol{G} \cdot \boldsymbol{x} + c'(x, y), \tag{3}$$

which separates c into the large-scale uniform gradient and a periodic flow-induced perturbation c'. Throwing (3) into (1) one obtains,

$$J(\psi, c') - \kappa \nabla^2 c' = -\boldsymbol{G} \cdot \boldsymbol{u}. \tag{4}$$

   (a) Using appropriate scaling parameters (l, κ, Ψ), show that eq. (4) takes the nondimensional form,

$$P\ J(\psi, c') - \nabla^2 c' = -\boldsymbol{\Gamma} \cdot \boldsymbol{u}. \tag{5}$$

   where P is the Péclet number, $P \equiv \Psi/\kappa$, and $\boldsymbol{\Gamma}$ is the nondimensional linear gradient.

(b) It is impossible to solve eq. (5) exactly. However approximate solutions in the limit $P \ll 1$ (diffusive limit) can be found by perturbation expansion techniques. Express the solution as $c' = c_0 + Pc_1 + P^2c_2 + \dots$ and solve for each order in P. Then compute,

$$\boldsymbol{F} = -\kappa\boldsymbol{\Gamma} + \langle \boldsymbol{u}c' \rangle, \tag{6}$$

where the average is carried over a periodic cell of dimension $l \times l$. The main point of this exercise is to show that $\boldsymbol{F}$ is directed down the large scale gradient $\boldsymbol{\Gamma}$. Calculate the solutions to second order in the small $P-$expansion (i.e. compute two terms in the expansion of the eddy diffusivity).

4. Look for a paper that describes how to compute neutral density and read it. Explain how neutral density is computed. Your answer cannot exceed half a page.

5. Download the data set ADELIE.mat from the class website. This is a short hydrographic section off the coast of the Antarctic Peninsula obtained in February 2007 by Andy Thompson, an oceanographer from Caltech. if you do nbot already have them, download the Matlab seawater routines. For this problem you will need to use (at least) the following seawater commands: sw_alpha, sw_beta, sw_dens, sw_dist, sw_pden, sw_ptmp. You should calculate the buoyancy frequency without using sw_bfrq.m.

   (a) From the *in situ* temperature and salinity data, plot sections of θ, σ_0, σ_2 and σ_4 as a function of pressure and distance along the section (Note that the sections are not equally spaced; be sure to account for this in your figures. Hint: use sw_dist).

   (b) Estimate the isopycnal slope at station 12 as a function of depth (be sure to explain how you made this calculation). How does the slope depend on the choice of reference pressure? Calculate the slope of the neutral density surfaces for the same station. How do they compare to the potential density slopes referenced at different levels?

12.801 General Ocean Circulation

Problem set 2: Wind driven circulation and single layer models

Due date: March 16, 2016

1. Doctor Kludge knows that the typical size of the Ekman pumping velocity in a subtropical gyre is $W = \pi \times 10$ meters per year. Kludge wants to estimate the order of magnitude of the gyre-scale North-South velocity, V. She looks at the mass conservation equation

$$u_x + v_y + w_z = 0.$$

and makes the following scale analysis

$$V \sim \frac{WL}{H} \tag{1}$$

where L is the meridional length-scale of V . Kludge plugs the numbers using $L = 5 \times 10^5$ meters and $H = 500$ meters as horizontal and vertical length scales. Mindlessly follow Kludge and get a numerical estimate for V using (1). Briefly discuss any additional considerations that Kludge, as a proud graduate of the Joint Program, should be aware of. Is Kludge's answer likely to be in the ballpark, or will the estimate of V change by an order of magnitude if one accounts for these factors? If you believe that Kludge is significantly in error make your own numerical estimate of V .

2. The meridional Sverdrup transport V_S is the depth-integrated transport over the entire water column. The meridional Ekman transport V_E is the velocity integrated over the frictional Ekman layer only. Splitting the flow field in its geostrophic and Ekman components, $\boldsymbol{u} = \boldsymbol{u}_G + \boldsymbol{u}_E$, and using the barotropic vorticity equation for the interior geostrophic flow and the known expressions for the meridional transports V_S and V_E, show that $V_S = V_G + V_E$. What is the ratio of V_E/V_G?

3. In class we showed that the barotropic streamfunction ψ in a flat ocean at leading order in Ro satisfies the equation,

$$\beta\psi_x = \sin \pi y/L - r\nabla^2\psi$$

where β is the planetary vorticity gradient, $\sin \pi y/L$ is the curl of the wind stress, L is the zonal extent of the ocean basin, and r is the drag coefficient. First write the full analytical solution to this equation, which satisfies the $\psi = 0$ (no normal flow) boundary conditions at $y = \pm L$ and $x = 0, \pm L$. Then expand your solution in the limit $r/L\beta \to 0$. Show that the solution in this limit reduces to the boundary layer solution we derived in class.

4. Munk suggested that lateral friction, instead of the bottom Ekman drag, may be an alternative way to close the wind-driven gyre problem. Show that the streamfunction equation for a barotropic wind-driven gyre with lateral friction satisfies the equation,

$$\beta\psi_x = \sin 2\pi y + \nu\nabla^4\psi,$$

where $0 < (x, y) < 1$. Assume that ν is small. (i) State the boundary conditions in terms of ψ and its derivatives. (ii) Give a complete account of the boundary layer which occurs in this problem; construct the solution using boundary layer theory.

5. Consider the barotropic quasi-geostrophic vorticity equation in shallow water:

$$\frac{D}{Dt}\left[\nabla^2\psi - F\psi + (f_0 + \beta y)\right] = 0,$$

where

$$F \equiv \frac{f_0^2 L^2}{g D_0}$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = \frac{\partial}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}.$$

Consider Rossby wave solutions on a mean zonal flow U =const.

(a) Derive an equation for the wave streamfunction.

(b) Assume a single plane wave solution and derive the dispersion relationship and the phase speed. Can you relate them to the dispersion relationship and phase speed of Rossby waves in a motionless ocean? What is the effect of the mean flow?

(c) Discuss the case of westerly flow $U = +1 > 0$ (i.e. flow coming from the west) and of easterly flow $U = -1 < 0$. Is a sattionary solution possible and, if so, when?

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Problem set 3: Wind driven circulation and single layer models

Due date: April 6, 2016

1. Read chapter 5 of John Marshall's notes on the wind driven circulation.
2. The equation for the depth-averaged flow with lateral friction in a square ocean (Munk's problem) is,

$$\beta \frac{\partial \psi}{\partial x} = W(y) + \nu \nabla^4 \psi$$

   where $W(y)$ is the wind stress curl, which does not vanish at the northern and southern boundaries.

   (a) What terms in the above equation are important in each boundary layer?

   (b) What terms in the above equation are important in the interior of the square ocean away from boundary layers?

   (c) Assuming that boundary layers are needed at all four coastal boundaries, what is the thickness of each one as a function of β, ν, and the length L of the sides of the square? Show how you derive the boundary layer thickness, without deriving the full solution to the problem, for:

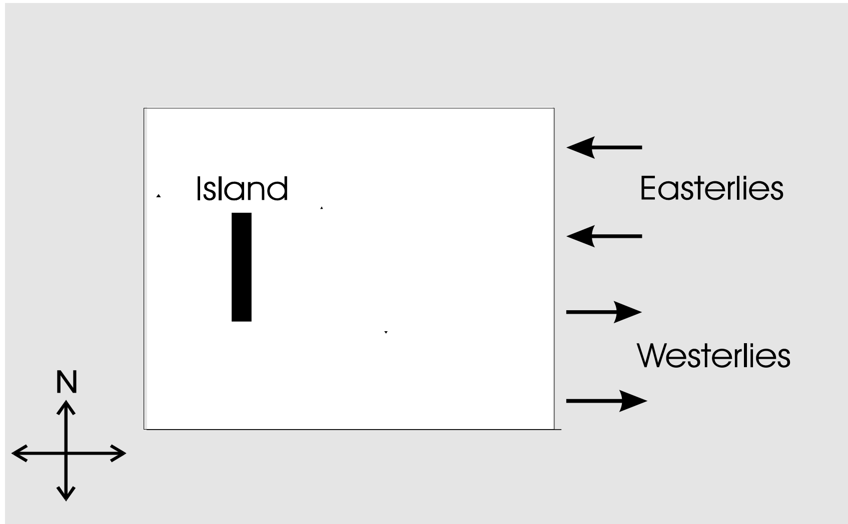
      i. thickness of the western boundary layer

      ii. thickness of the eastern boundary layer

      iii. thickness of the northern boundary layer

      iv. thickness of the southern boundary layer

3. Use Sverdrup theory and the idea that only western boundary currents are allowed, to sketch the pattern of ocean currents you would expect to observe in the basin sketched overleaf in which there is an island. Assume a wind pattern of the form sketched in the diagram.



4. Consider the Atlantic Ocean to be a rectangular basin, centered on 35°N, of longitudinal width $L_x = 5000$km and latitudinal width $L_y = 3000$km. The ocean is subjected to a zonal wind stress of the form

$$\tau_x(y) = -\tau_s \cos\left(\frac{\pi y}{L_y}\right); \qquad \tau_y(y) = 0; \tag{1}$$

where $\tau_s = 0.1\text{Nm}^{-2}$. You may assume a constant value of $\beta = df/dy$ appropriate to 35°N, that the ocean has uniform density 1000 kg m^{-3} and adopt local Cartesian geometry.

   (a) From the Sverdrup relation determine the magnitude and spatial distribution of the depth-integrated meridional flow velocity in the interior of the ocean.

   (b) Using the depth-integrated continuity equation, and assuming no zonal flow at the eastern boundary of the ocean, determine the magnitude and spatial distribution of the depth-integrated zonal flow in the interior.

   (c) If the return flow at the western boundary is confined to a width of 100 km, determine the depth-integrated flow in this boundary current.

   (d) If the flow is confined to the top 500m of the ocean (and is uniform with depth in this layer), determine the northward components of flow velocity in the interior, and in the western boundary current.

   (e) Compute and sketch the pattern of Ekman pumping implied by the idealized wind pattern, Eq. (1).

5. From your answer to question 3, determine the net volume flux at 35°N (the volume of water crossing this latitude in units of Sverdrups: Sv = 10^6m^3 s^{-1}).

   (a) for the entire ocean excluding the western boundary current

(b) for the western boundary current only.

(c) Assume again that the flow is confined to the top 500m of the ocean. Determine the volume of the top 500m of the ocean and, by dividing this number by the volume flux you calculated in part (a), come up with a time scale. Discuss what this time scale means.

(d) Assume now that the water in the western boundary current has a mean temperature of 20°C, while the rest of the ocean has a mean temperature of 5°C. Show that H_{ocean}, the net flux of heat across 35°N, is

$$H_{ocean} = \rho_{ref} c_p V \Delta T,$$

where V is the volume flux you calculated in part (c), and ΔT is the temperature difference between water in the ocean interior and in the western boundary current. Given that Earth's energy balance requires a poleward heat flux of around 5×10^{15}W, estimate and discuss what contribution the Atlantic Ocean makes to this flux.

6. For this exercise you will need to run Bryan's solver of the one layer QG equations. Setup a wind-driven gyre problem in a closed rectangular box with the following parameters: $\tau^{(x)} = 0.2 \sin(\pi y/L)$ N m^{-2}, $H_0 = 1000$ m, $f_0 = 10^{-4}$ s^{-1}, $\beta = 2 \times 10^{-11}$ m^{-1}s^{-1}, $r = 5 \times 10^{-6}$ s^{-1}, $L = 5000$ km. These standard set of parameters are set in the "*qg1l_basin.jl*" provided with the homework–please download the latest version. You may notice that Bryan added a small lateral friction to keep the solution stable. Ignore this in your discussion of the solutions.

(a) Run a simulation with the suggested parameters for friction and wind stress. Wait until the simulation comes to an equilibrium. Look at the streamfunction. Describe the solution you obtain. Is the interior solution consistent with Sverdrup's theory for the wind stress you are using? If so, explain how the interior transport reflects the wind stress pattern. Is there a frictional boundary current? On what boundary? What sets its thickness?

(b) Change the value of β to $2{\times}10^{-13}$m^{-1}s^{-1} leaving all other parameters unchanged. How does the solution change? Discuss these changes in terms of what you know about Stommel's model of the wind-driven gyre.

(c) Set β back to its midlatitude typical value of $2{\times}10^{-11}$m^{-1}s^{-1}. Decrease the frictional parameter to $3.0{\times}10^{-7}$ s^{-1}. Wait until the solution equilibrates. How did the solution change compared to case (a)? Are there frictional or inertial boundary layers? Where?

12.801 General Ocean Circulation

Problem set 4: Wind driven circulation and multi layer models

Due date: April 25, 2014

1. **Gyre spinup.** Consider the planetary-scale two-layer QG model. Suppose that the two layers have equal depth, $H_1 = H_2 = H$, and that the reduced gravity is $g' \ll g$. The QG equation is

$$\begin{aligned} \partial_t q_1 + J(\psi_1, q_1) &= \frac{f_0 w_0}{H} \sin\left(\frac{\pi y}{L}\right) \mathcal{H}(t), \\ \partial_t q_2 + J(\psi_2, q_2) &= 0, \end{aligned}$$

where the potential vorticities are

$$q_n = \beta y + F(-1)^n(\psi_1 - \psi_2), \qquad F \equiv f_0^2/g'H.$$

Notice that $\mathcal{H}(t)$ is the step-function, not the depth of the ocean. The domain is a rectangle $0 < x < L$ and $-L < y < L$. The wind-stress is switched on suddenly at $t = 0$ and before that $\psi_n = 0$.

(a) Show that the barotropic mode

$$\psi_B \equiv \frac{\psi_1 + \psi_2}{2},$$

adjusts instantly to Sverdrup balance:

$$\psi_B(x, y, t) = -\Psi\left(1 - \frac{x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \mathcal{H}(t).$$

Determine Ψ in terms of L, w_0, β etcetera. Instant adjustment is obviously an approximation; say a little about the time-scale over which the barotropic mode comes into Sverdrup balance (twenty words or so).

(b) Eliminate ψ_1 and obtain a single equation for the lower layer flow, ψ_2:

$$\partial_t \psi_2 - J(cy + \psi_B, \psi_2) = \partial_t \psi_B.$$

Give the constant c in terms of F and β. Physically interpret c in terms of the properties of the baroclinic mode.

(c) Draw the lower-layer geostrophic contours, $cy + \psi_B$, using various values of the nondimensional ratio cL/Ψ. Using these figures, discuss the time evolution of ψ_2. Does this initial value problem unambiguously determine the vertical structure of the Sverdrup flow i.e. the partitioning of ψ_B into ψ_1 and ψ_2?

2. **Ventilated thermocline**. Consider the ventilated thermocline model described in class. The domain is contained in $0 < x < L$ and $0 < y < L$, the wind forcing is described by $\tau = -\tau_0 \cos(\pi y/L)$, the lower layer outcrops at a latitude $y_2 = 0.8L$ and the depth of layer 2 is equal to H_e on the eastern boundary north of y_2.

   (a) Find an expression for the curve that marks the boundary of the shadow zone. You can leave your answer in terms of the function $D_0^2(x, y)$ introduced in class. (Hint: You should find an expression for $D_0^2(x = x_S(y), y)$, where x_S is the longitude position of the shadow zone boundary and is a function of latitude).

   (b) Imagine there is a third, motionless, layer below layer two. Its depth on the eastern boundary is equal to layer 2, i.e. on the eastern boundary $h_2+h_3 = 2H_e$. What is the potential vorticity distribution in this lowermost layer. Indicate where the meridional PV gradient changes sign. What does this physically correspond to?

3. **Baroclinic instability.** Show that the two-layer QG equations on a β-plane and a bottom topography at $z = -H_2 + h_b$ are:

$$\partial_t q_1 + J(\psi_1, q_1) = 0, \qquad q_1 = \nabla^2\psi_1 + \beta y + F_1(\psi_2 - \psi_1), \tag{1}$$

$$\partial_t q_2 + J(\psi_1, q_2) = 0, \qquad q_2 = \nabla^2\psi_2 + \beta y + F_2(\psi_1 - \psi_1) + f_0\frac{h_b}{H_2}, \tag{2}$$

$$\tag{3}$$

   where $F_1 = f_0^2/g'H_1$, $F_2 = f_0^2/g'H_2$ are the squared inverse deformation wave numbers and h_b. Consider a basic state with a zonal constant flow U_1 in the upper layer and a zonal constant flow U_2 in there lower layer.

   (a) Set $h_b = 0$ and assume $(H_1, H_2) = (1000, 3000)$ m and $(U_1, U_2) = (0.1, 0)$ m s^{-1}, representative values for the Antarctic Circumpolar Current (ACC). Is the ACC baroclinically stable?

   (b) Set $h_b = sy$. Determine what is the minimum shear, $U_1 - U_2$, for which the mean flow goes baroclinically unstable. Does the bottom slope s increase or decrease the minimum shear compared to the flat bottom problem? How large a slope do you need to stabilize the ACC based on the parameters given above? Is this a slope you may expect to find in the bottom of the Southern Ocean on scales of thousand of kilometers?