Marshall *vs.* Walras on Equilibrium and Disequilibrium

Franco Donzelli[1]
Department of Economics, Business and Statistics,University of Milano

In all introductory and intermediate textbooks in microeconomics, price theory is at first developed in a partial equilibrium framework which, though invariably Marshallian from the point of view of the graphical apparatus, is almost always Walrasian from the point of view of the analysis. In discussing the well-known Marshallian graphical treatment of the equilibrium problem of a single, isolated, assumedly "competitive" market for a producible consumers' good, a few diligent textbook writers[2] occasionally point out that placing the price and quantity variables on the ordinate and abscissa axes, respectively, is somewhat incongruous with the role played by the same two variables in the analytical development of the theory: for, from a Walrasian perspective, price is the independent variable, while quantity is the dependent one (whatever this may mean). Such incongruousness is sometimes historically justified by recalling that price had indeed been regarded as the independent variable by Marshall, to whom the demand-and-supply graphical apparatus can be traced back; but this suggestion, never elaborated upon, hovers about as a mysterious reference to a by now forgotten past. As a matter of fact, in the partial equilibrium framework of an isolated market, to which microeconomic primers confine most of their discussions of price theory, almost all methodological, epistemological, as well as analytical distinctions between Marshall's and Walras's approaches are skipped over. Moreover, when intermediate and advanced microeconomic textbooks eventually deal with the issue of price formation in a multi-market framework, general equilibrium theory is invariably presented as the natural extension of partial equilibrium analysis, as if Walrasian and Marshallian approaches to price theory only differed in scope and intended applications, being otherwise essentially similar in their foundations and results[3].

In this paper we want to oppose the received view on the basic equivalence of the two traditional approaches to price theory. Specifically, we want to show that Marshall's analysis of the equilibration process and his related interpretation of the equilibrium concept are essentially different from, and irreducible to, Walras's analysis and interpretation. Further, we want to show that the patent difference in scope of their respective theories (that is, partial *vs*. general analysis), far from being an accidental outcome of history or the innocuous consequence of the idiosyncratic preferences of the two economists, is in effect the unavoidable and irremediable by-product of their different analytical assumptions and explanatory aims.

To this end, we shall first identify a common ground for our discussion, that is, a model economy that, being explicitly examined by both authors in their respective writings, will allow us to contrast their analyses and to precisely single out what distinguishes them from one another. Such model economy is represented by the two-commodity, pure-exchange economy which is dealt with by both Walras and Marshall right at the beginning of their respective expositions of price theory, under the assumption that there exists a fixed finite number of traders in the economy. Yet, while Walras's analysis is entirely developed under the assumption of an arbitrary finite number of traders (greater than or equal to two), in Marshall's case one finds a distinction between two sorts of economies: the former, called by Marshall a "barter" economy and more recently referred to as an "Edgeworth Box" economy, is an economy where the number

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[2] Baumol ([1977], p. 179) and Chiang ([2005], p. 32).
[3] Varian ([1992], p. 105-108) and Mas-Colell, Whinston, and Green ([1995], Ch. 10, p. 311-349).

of traders is exactly equal to two; the latter, analyzed by Marshall in what he calls his "temporary" or "market-day equilibrium" model, is instead an economy where the number of traders is strictly greater than two, but otherwise arbitrary.

Coming then to Walras, after introducing his three basic assumptions about the trading process (that is, the so-called "Law of One Price", the "Perfect Competition" assumption, and the "No Trade out of Equilibrium" assumption), we shall show that such assumptions have a twofold, ambiguous effect on his theorizing: for, on the one hand, they force him to confine his attention to a purely virtual equilibration process in "logical" time; on the other, however, they allow him to identify a well-defined equilibrium solution, of the "instantaneous" type, under relatively mild and general assumptions on the traders' characteristics (specifically, their utility functions). Further, under the same or similar assumptions, Walras's analysis can be easily extended to exchange, and even production, economies with any finite number of commodities.

As to Marshall, instead, we shall show that he does not make any one of the three assumptions that we have seen to characterize Walras's approach to price theory: this is due to the fact that, unlike Walras, Marshall deliberately aims at analyzing an equilibration process in "real" time, where different transactions may take place at different prices at the same time, traders do not take prices as given and, of course, trades may occur at out-of-equilibrium prices. Under these conditions, however, in order to make the trading process converge to a determinate solution, it is necessary to make much stronger assumptions about the traders' characteristics (specifically, their utility functions): this is the reason why, in Marshall's analysis, the assumption of a "constant marginal utility of money" comes into the picture. Under this assumption, Marshall can indeed prove that, in an Edgeworth Box economy with a commodity proper and a money-commodity, the trading process converges to a determinate equilibrium (namely, to determinate equilibrium values of both the money price and the total quantity traded of the commodity proper). As we shall see, however, the result is less neat than Marshall probably expected or hoped for, when the above analysis is generalized to a pure-exchange, two-commodity, monetary economy with more than two traders. Yet, there is a further limitation which is much more important for our present purposes: precisely, there is no way to formally extend Marshall's analysis and results to a multi-commodity, multi-market economy, so that partial equilibrium analysis becomes the unavoidable stigma of Marshall's approach.

The remainder of the paper is organized as follows. In section 2 we describe the model economy providing the common ground for our analysis, namely, the pure-exchange, two-commodity economy with a finite number of traders, greater than or equal to two, which underlies both Marshall's and Walras's initial theorizing about price theory. Section 3 is devoted to Walras's analysis. In this case, due to the relatively formalized expository style adopted by Walras himself, it proves convenient to keep the formal statement of the theory disconnected from the informal discussion of both its interpretation and the textual evidence supporting it: hence, in subsection 3.1, we start by stating the three basic assumptions about the trading process on which Walras's analysis (in its final form) is based; in subsection 3.2 we formally state Walras's equilibrium model of a pure-exchange, two-commodity economy with an arbitrary finite number of traders (if this number were equal to two, the economy would boil down to an Edgeworth Box economy, but this restriction is unnecessary in Walras's case); subsection 3.3 deals with interpretative and hermeneutical issues; finally, the possibility of further extending Walras's model to multi-commodity exchange and production economies is discussed in subsection 3.4. Section 4 is devoted to Marshall's analysis. Due to Marshall's peculiar style of exposition, which is eminently literary and unformalized, it seems preferable, in this case, to take a different route from that followed in examining Walras's approach: precisely, starting from a hermeneutical discussion of Marshall's informal account of his own approach, we shall strive to jointly reconstruct both the implicit formal apparatus and the associated interpretation of Marshall's theory. Specifically, in subsection 4.1, we discuss Marshall's basic assumptions about the trading process; in subsection 4.2 we formalize Marshall's model of an Edgeworth Box economy, one version of which deals with an economy where one of the two commodities is a money-commodity; then, in subsection 4.3, we discuss Marshall's so-called "temporary equilibrium" model, which can be viewed as a generalization of the model of an Edgeworth Box economy with a money-commodity, allowing for a finite number of traders greater than two; finally, the possibility of further extending Marshall's "temporary equilibrium" model to multi-commodity exchange and production economies is discussed in subsection 4.4. Section 5 compares the two approaches and concludes.

II. A common ground for the analysis: the pure-exchange, two-commodity economy

Let us consider Walras's and Marshall's main theoretical works, namely, Walras's *Eléments d'économie politique pure*[4] and Marshall's *Principles of Economics*[5]. While most chapters of the *Eléments* are explicitly devoted to competitive price theory (no less than thirty one Lessons out of the forty two composing the fourth and fifth editions of the *Eléments* deal with that topic), the same is not true of the *Principles*: since its second edition, in fact, Marshall's treatise consists of six Books, of which only one (Book V, on "*The General Relations of Demand, Supply, and Value*") is entirely devoted to price theory. But, apart from the different quantitative emphasis the two books place on price theory, they so widely differ in their qualitative treatment of that subject that a quick reader might easily be led to despair of the reasonableness or fruitfulness of any formal comparison between the two approaches. Yet, at a closer inspection, a well-defined set of theoretical issues can be identified that represent the common starting point for both Walras's and Marshall's inquiries into the field of price theory. Such common starting point consists in the problem of the determination of equilibrium prices and quantities in a pure-exchange, two-commodity economy: Walras deals with that problem in Part II, Lessons 5 to 10, of the fourth and fifth editions of the *Eléments* ([1954], p. 83-152)[6]; Marshall deals with it in Chapter II of Book V and in Appendix F of the *Principles* ([1961a], p. 331-336 and 791-793)[7]. Even if, from a quantitative point of view, the theory of the determination of equilibrium prices and quantities in a pure-exchange, two-commodity economy represents only a small part of

[4]During Walras' lifetime, four successive editions of the *Eléments* were sent to press: the first one, subdivided into two installments, appeared in 1874 and 1877; the second, third, and fourth editions, instead, were each published as a unitary volume in 1889, 1896, and 1900, respectively. There was also a posthumous edition, arranged by Walras himself before his death and almost identical with the fourth, which was published in 1926; this edition, known in the past as the "quatrième édition définitive", is now more simply indicated as the fifth edition. In the following we shall chiefly refer to Jaffé's English edition (Walras [1954]), which is based on the fifth edition of the *Eléments*. Occasionally, however, it will be necessary to mention or quote one specific edition of that book. In that case, we shall refer to the *variorum* edition of the *Eléments*, contained in vol. VIII of the *Oeuvres économiques complètes d'Auguste et de Léon Walras* (Walras [1988]), which allows easy comparisons among the texts of the various editions.

[5] During Marshall's lifetime, eight successive editions of the *Principles of Economics* were published, from 1890 to 1920. In the following we shall refer to the so-called Ninth (Variorum) Edition, published in 1961 with annotations by C.W. Guillebaud. This edition consists of two volumes: *Volume I. Text*, containing the text of the eighth edition of the *Principles* (Marshall [1961a]), and *Volume II. Notes*, containing both the collation notes and other editorial notes by Guillebaud (Marshall [1961b]).

[6]Lessons 5 to 10 immediately follow the introductory Part I of the *Eléments*, being therefore the first Lessons of that book devoted to price theory in the strict sense.

[7]While "Appendix F. Barter" deals with a pure exchange, two-commodity, two-trader economy, chapter II of Book V, "*Temporary equilibrium of demand and supply*", deals with a pure exchange economy with two commodities, one of which is money, and an arbitrary finite number of traders. It should be noted that in the first four editions of the *Principles* the subject-matter of what would later become, since the fifth edition, "Appendix F. Barter" was placed at the end of Book V, chapter II, and was entitled "*A Note on Barter*" ([1961b], p. 790). The strict logical connection between the contents of chapter II of Book V and Appendix F, which comes out clearly from a sequential reading of the two physically disconnected passages in the fifth and following editions of the *Principles*, was made even more evident by the physical contiguity of the two passages in the previous editions of that book. Anyhow, even in the last three editions of the *Principles*, the link between the two disconnected sections is made explicit by a reference to Appendix F in the last paragraph of chapter II of Book V (Marshall [1961a], p. 336).

Walras's overall competitive equilibrium theory, as put forward in the *Eléments*, and an even smaller part of Marshall's overall theory of market equilibrium, as developed in Book V of the last three editions of the *Principles*, yet such theory plays a fundamental role in either author's theoretical construction, for in either case it is the cornerstone on which the whole building is erected[8]. Anyhow, since the problem of equilibrium price determination in a pure-exchange, two-commodity economy is the only problem which is formally discussed by both authors in their respective treatises with the help of similar analytical tools, any comparison between the two authors, as far as price theory is concerned, cannot but start from the analysis of their respective models of the simplified economy under discussion.

Let us then consider a pure-exchange economy with $L = 2$ commodities, denoted by $l = 1,2$, and I consumers-traders (henceforth indifferently referred to as either consumers or traders), denoted by $i = 1,...,I$, with $I \geq 2$. Each consumer i is characterized by a consumption set $X_i = \{x_i \equiv (x_{1i}, x_{2i})\} = \Re^2_+$, a preference relation $\geq_i$ on X_i, and endowments $\omega_i \equiv (\omega_{1i}, \omega_{2i}) \in \Re^2_+ \setminus \{0\}$. Let $x = (x_1,..., x_I) \in X = \times_{i=1}^I X_i \subset \Re^{2I}_+$ be an allocation; $\bar{\omega} = \sum_{i=1}^I \omega_i \in \Re^2_{++}$ be the aggregate endowments; $A^{2\times I}_{pe} = \left\{ x \in X \mid \sum_{i=1}^I x_i = \bar{\omega} \right\}$ be the set of feasible, non-wasteful allocations. Specifically, in accordance with Walras's and Marshall's original assumptions, let us assume consumer i's preferences to be represented by a cardinal utility function $u_i : X_i \to \Re$, which, for all i, is supposed to be both additively separable, that is,

$$u_i(x_i) = v_{1i}(x_{1i}) + v_{2i}(x_{2i}), \forall x_i \in X_i$$ [9]

and twice continuously differentiable, with

$$\nabla u_i(x_i) = \left(\frac{\partial u_i(x_i)}{\partial x_{1i}}, \frac{\partial u_i(x_i)}{\partial x_{2i}} \right) = (v'_{1i}(x_{1i}), v'_{2i}(x_{2i})) >> 0, \forall x_i \in X_i,$$

$$\left(\frac{\partial^2 u_i(x_i)}{\partial x^2_{1i}}, \frac{\partial^2 u_i(x_i)}{\partial x^2_{2i}} \right) = (v''_{1i}(x_{1i}), v''_{2i}(x_{2i})) < 0, \forall x_i \in X_i,$$

where $>> 0$ in the first inequality means that the first-order partial derivatives of consumer i's utility function, i.e., i's marginal utility functions, are strictly positive, while < 0 in the second inequality means that the pure second-order partial derivatives are non-positive[10].

Such an economy will be denoted by $\varepsilon^{2\times I}_{pe} = \{(X_i, u_i(\cdot), \omega_i)_{i=1}^I\}$ in the following. When $I = 2$, the pure-exchange, two-commodity, two-consumer economy $\varepsilon^{2\times 2}_{pe} = \left\{ (\Re^2_+, u_i(\cdot), \omega_i)_{i=1}^2 \right\}$ will be called an Edgeworth Box economy and denoted by ε_{EB} in the following.

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[8]Walras is ready to acknowledge the central role played in the development of his system of thought by his analysis of the equilibrium determination problem in a pure-exchange, two-commodity economy ([1954], p. 143). Marshall, on the contrary, is reluctant to openly ascribe a significant role to his pure-exchange models (that is, the "barter model" and the "temporary equilibrium" one). But, in spite of Marshall's public propensity to play down the relevance of such models in his theorizing, we shall show that the theoretical solutions adopted therein end up by crucially affecting his entire theoretical system (what, incidentally, is recognized by Marshall himself in private correspondence, as we shall see in Section 4.4 below).

[9] As we shall see, the cardinality and additive separability assumptions concerning the consumers' utility functions play a completely different role in Walras' and Marshall's theoretical systems: for while they can easily be disposed of in Walras' case, they cannot instead be relaxed in Marshall's case without jeopardizing his whole theoretical construction

[10]The above restrictions on the first- and second-order partial derivatives of the utility functions need some further qualifications, which will be provided in subsections 3.2 and 4.2, concerning Walras and Marshall, respectively.

Given a pure-exchange, two-commodity, I-consumer economy $\varepsilon \in \mathbb{E}_{pe}^{2\times I}$, for all i and all $x_i \in X_i$, let

$$MRS_{21}^i(x_i) \equiv \left|\frac{dx_{2i}}{dx_{1i}}\right|_{u_i(x_i+dx_i)=u_i(x_i)} = \frac{\dfrac{\partial u_i(x_i)}{\partial x_{1i}}}{\dfrac{\partial u_i(x_i)}{\partial x_{2i}}} = \frac{v_{1i}'(x_{1i})}{v_{2i}'(x_{2i})}$$

be consumer i's marginal rate of substitution of commodity 2 for commodity 1 when i's consumption is x_i: namely, $MRS_{21}^i(x_i)$ is the quantity of commodity 2 that consumer i would be willing to exchange for one unit of commodity 1 at the margin, in order to keep his utility unchanged at the original level $u_i(x_i)$.Let $z_i(x_i) \equiv (z_{1i}, z_{2i})(x_i) \equiv x_i - \omega_i \equiv (x_{1i} - \omega_{1i}, x_{2i} - \omega_{2i}) \in \Re^2$ be consumer i's net demand, when his consumption is x_i. Consumer i's net demand for commodity l, $z_{li}(x_i)$, can be either positive, in which case $z_{li}(x_i)$ is called consumer i's net demand proper for commodity l and consumer i is said to be a net buyer of that commodity, or negative, in which case $|z_{li}(x_i)|$ is called consumer i's net supply of commodity l and consumer i is said to be a net seller of that commodity.

Now, let us suppose that consumer i can trade commodity 2 for commodity 1 either on the market, or through bilateral bargains, or according to any other suitably specified voluntary exchange technology. When consumer i's consumption is x_i, if the marginal rate at which i can trade commodity 2 for commodity 1, that is $-\frac{dx_{2i}}{dx_{1i}} = \left|\frac{dx_{2i}}{dx_{1i}}\right|$, is exactly equal to $MRS_{21}^i(x_i)$, then i's utility is unaffected by a marginal trade of commodity 2 for commodity 1, irrespective of whether i is a net buyer or seller of commodity 1; for in that case:

$$du_i(x_i) = \nabla u_i(x_i)dx_i = \frac{\partial u_i(x_i)}{\partial x_{1i}}dx_{1i} + \frac{\partial u_i(x_i)}{\partial x_{2i}}dx_{2i} = v_{1i}'(x_{1i})dx_{1i} + v_{2i}'(x_{2i})dx_{2i} = 0.$$

On the contrary, consumer i's utility increases if the marginal rate of exchange of commodity 2 for commodity 1 is less (resp., greater) than $MRS_{21}^i(x_i)$, provided that i is a net buyer (resp., seller) of commodity 1. Hence $MRS_{21}^i(x_i)$ can also be interpreted as the maximum (resp., minimum) quantity of commodity 2 that a utility maximizing buyer i (resp., seller i) of commodity 1 is willing to pay (resp., to receive) at the margin in exchange for one unit of commodity 1, when i's consumption is x_i. By using an expression which is currently employed in the literature in a related context, we can summarize the above interpretation of the marginal rate of substitution by saying that $MRS_{21}^i(x_i)$ represents consumer i's "reservation price" of commodity 1 in terms of commodity 2, when i's consumption is x_i. (Though the expression "reservation price" can be indifferently employed irrespective of whether consumer i is a buyer or a seller, its specific meaning depends of course on the nature of the trade that i is willing to carry out.)

Both Walras and Marshall do not exactly employ the conceptual apparatus developed above. In particular, they both ignore the notion of marginal rate of substitution or, for that matter, that of reservation price. Yet, they do know and systematically employ the notion of marginal utility of commodity l for consumer i, which, under the stated assumptions on the properties of the utility functions, turns out to be a function of the quantity of commodity l only. Moreover, though not explicitly discussing the notion of marginal rate of substitution as such, they do implicitly make use of it in their analyses, since they compute the ratio of any two marginal utility functions and examine the role of such ratio in explaining the choices of consumers. Hence one can legitimately say that the above conceptual apparatus, though slightly more general than that originally employed by Walras or Marshall, provides a common foundation on which to erect either economist's demand-and-supply analysis (subject to the qualifications concerning the first- and second-order partial derivatives of the utility functions, to be discussed in the next sections). Any further development of either Walras's or Marshall's approach, however, requires further assumptions, which are specific to either economist. To such specific assumptions we now turn our attention.

II. Walras's approach

As anticipated in the introductory section, in Walras's case it is convenient to put forward the formal model of a pure-exchange, two-commodity economy first, postponing all interpretative issues to a later subsection.

1. Three basic assumptions about the trading process

To begin with, let us state three assumptions which, as we shall see, underlie not only the simple model with which we are exclusively concerned here, but indeed all of Walras's equilibrium models, provided that they are taken in their final form (that is, in the form given to them in the fourth edition of the *Eléments* [1900]). In order to make the understanding of Walras's approach to price theory easier, the basic assumptions about the trading process are separately stated in the following, even if they are obviously interrelated and often confused, occasionally by Walras himself, or jointly formulated in the literature.

Assumption 1. (**"Law of One Price"**)
At each instant of the trading process, a price is quoted in the market for each commodity. Moreover, if any transaction concerning a given commodity takes place at any instant of the trading process, then it takes place at the price quoted at that instant.

Assumption 2. (**"Perfect Competition"**)
All traders behave competitively, that is, they take prices as given parameters in making their optimizing choices.

Assumption 3. ("**No Trade out of Equilibrium"**)
No transaction concerning any commodity is allowed to take place out of equilibrium.

The wording of the above assumptions has been carefully chosen in order to make their statement consistent with Walras's original discussion, ambiguities not excepted. The exact meaning of the assumptions cannot be explained without first defining the undefined terms appearing therein. The required definitions will be given in the next subsection, with specific reference to the model of a pure-exchange, two-commodity economy, while a general discussion of the assumptions is deferred to subsections 3.3 and 3.4 below.

2. Walras's model of a pure-exchange, two-commodity economy

Let us consider a pure-exchange, two-commodity economy $\varepsilon_{pe}^{2\times I} = \{(X_i, u_i(\cdot), \omega_i)_{i=1}^{I}\}$, where the consumers' characteristics satisfy all the assumptions made in section 2, with the further restriction that, for all i, the second-order pure partial derivatives of the utility functions be strictly negative , that is

$$\left(\frac{\partial^2 u_i(x_i)}{\partial x_{1i}^2}, \frac{\partial^2 u_i(x_i)}{\partial x_{2i}^2}\right) = (v_{1i}''(x_{1i}), v_{2i}''(x_{2i})) << 0, \forall x_i \in X_i.$$

The assumptions on the signs of the partial derivatives of the consumers' utility functions that we have adopted here are in effect more demanding than Walras's original ones: for Walras typically assumes the marginal utility of commodity l to go to zero for $x_{li} < \infty$ ([1954], p. 117). On the contrary, with a view to simplifying our discussion, we assume here the marginal utilities of both commodities to be strictly positive and monotonically decreasing over each consumer's entire consumption set: this assumption, allows us to dodge all boundary problems and obtain well-defined demand and excess-demand functions, can anyhow be dispensed with, at the cost of complicating somewhat the analysis.

Let $p = (p_1, p_2) \in \Re^2_{++}$ be the price system, where prices are expressed in terms of units of account. The assumed positivity of prices is justified by the assumption of strong monotonicity of consumers' preferences. In view of Assumption 1, one ought to specify the instant of the trading process at which a given price system is quoted. Yet, since traders' choices necessarily refer to the same instant as the quoted prices, while the data (consumption sets, preferences, endowments) are assumed to be invariant over the exchange process ([1954], p. 117, 146), all the variables appearing in the following equations, which formalize the equilibrium determination problem ("economic *statics*", in Walras's words), would invariably refer to one and the same instant, namely, that instant at which prices are supposed to be quoted. This, however, makes the dating of the variables superfluous. Hence, following Walras's own lead in this respect, we avoid qualifying the price system with any time subscript referring to the evolution of the trading process: we know that such process evolves over time, but we do not need, at this stage, to make such evolution explicit[11]. Finally, under Assumptions 1 and 2, consumers' optimizing choices turn out to be homogeneous of degree zero in prices, as we shall see in a moment. But this implies that the price system can be normalized without any effect on consumers' choices. With just two commodities, we only need one relative price, namely, $p_{12} \equiv \frac{p_1}{p_2} \equiv p_{21}^{-1}$.

Focusing on this relative price is tantamount to normalizing the price system by taking commodity 2 as the numeraire, which in turn means setting $p_2 \equiv 1$. Under the stated assumptions, solving the constrained utility maximization problem for competitive consumer i results into the following two-equation system:

$$\frac{\dfrac{\partial u_i(x_{1i}, x_{2i})}{\partial x_{1i}}}{\dfrac{\partial u_i(x_{1i}, x_{2i})}{\partial x_{2i}}} = \frac{v_{1i}'(x_{1i})}{v_{2i}'(x_{2i})} = p_{12} \tag{1}$$

$$p_{12}x_{1i} + x_{2i} = p_{12}\omega_{1i} + \omega_{2i}, \tag{2}$$

from which one gets consumer i's Walrasian direct demand and excess demand functions, $x_i(p_{12}, \omega_i)$ and $z_i(p_{12}, \omega_i) \equiv x_i(p_{12}, \omega_i) - \omega_i$, respectively, for $i = 1, ..., I$.

Now, under Assumptions 1 and 2, aggregating individual demand and excess demand functions is immediate: for, since all consumers receive the same price signals (by Assumption 1), which they take as given parameters (by Assumption 2), the individual demand and excess demand functions always depend on the same variables and can consequently be summed over

[11] As Walras himself puts it, in introducing equations similar to those discussed in the sequel of this subsection : "I am assuming that, during this interval, the utility, both extensive and intensive, remains *fixed* for each party, which makes it possible for me to include time implicitly in the expression of utility. Were this not the case and had I supposed utility to be a *variable* functionally related to time, then time would have had to figure explicitly in the problem. And we should then have passed from economic *statics* to economic *dynamics*" ([1954], p. 117; Walras' italics).

all consumers. Hence, letting $z(p_{12},\omega)\equiv\sum_{i=1}^{I}z_i(p_{12},\omega_i)\equiv\sum_{i=1}^{I}x_i(p_{12},\omega_i)-\omega_i$ be the Walrasian aggregate excess demand function, where $\omega=(\omega_1,...,\omega_I)$, we obtain the market clearing conditions:

$$z_1(p_{12}^W,\omega)=0 \qquad (3')$$

and

$$z_2(p_{12}^W,\omega)=0, \qquad (3'')$$

where p_{12}^W denotes a Walrasian equilibrium price of commodity 1 in terms of commodity 2.

From the budget constraint equations (2), by rearranging terms and summing across consumers, we get the so-called Walras' Law, that is:

$$\sum_{i=1}^{I}\left[p_{12}z_{1i}(p_{12},\omega_i)+z_{2i}(p_{12},\omega_i)\right]=p_{12}z_1(p_{12},\omega)+z_2(p_{12},\omega)=0,\forall p_{12}\geq 0.$$

Since, due to Walras' Law, equation $(3'')$ is necessarily satisfied when equation $(3')$ holds ([1954], p.139), we can focus attention on the latter equation only. Under the stated assumptions, equation $(3')$ has at least one solution, which however needs not be unique. Each solution yields a Walrasian equilibrium price of commodity 1 in terms of commodity 2, p_{12}^W, to which a corresponding Walrasian equilibrium allocation, $x(p_{12}^W)=(x_1(p_{12}^W),...,x_i(p_{12}^W),...,x_I(p_{12}^W))$, is associated.

3. Walras's model: textual evidence and interpretation

Economists are so accustomed to regarding the model put forward in the previous subsection as typically Walrasian that many, or even most, of them may deem it otiose to inquire whether or not the model, as well as the assumptions on which it rests, can indeed be traced back to Walras's *Eléments*. Yet this question is by no means trivial: answering it will prove much more complicate than it might appear at first sight. Right at the beginning of Lesson 5 of the *Eléments*, where Walras starts his discussion of the "problem of the exchange of two commodities for each other", one finds a long illustrative passage, where the functioning of a real-word competitive market, the market for the so-called "*3 per cent French Rentes*", is described in great detail. This example is obviously meant to provide a gradual introduction to the more formal examination of the problem at issue, to be developed in the following pages. Precisely owing to its informal character, however, Walras's introductory discussion of the functioning of a real-world market discloses a number of conceptual difficulties, which are instead concealed under the more cautious language of formal analysis. Hence it may be useful to start from the securities example[12]: "*Let us take, for example, trading in 3 per cent French Rentes on the Paris Stock Exchange and confine our attention to these operations alone. The three per cent, as they are called, are quoted at 60 francs. [...] We shall apply the term effective offer to any offer made, in this way, of a definite amount of a commodity at a definite price. [...] We shall apply the term effective demand to any such demand for a definite amount of a commodity at a definite price. We have now to make three suppositions according as the demand is equal to, greater than, or less than the offer. First supposition. The quantity demanded at 60 francs is equal to the quantity offered at this same price. [...] The rate of 60 francs is maintained. The market is in a stationary state or equilibrium. Second supposition. The brokers with orders to buy can no longer find*

[12]The following quotation in the text is drawn from the English edition of the *Eléments* ([1954], p. 84-85). However, for reasons that will become apparent later in this subsection, we have reproduced the passage as it originally appeared in the first edition of the *Eléments* (apart from the English translation, of course), suppressing a few words inserted by Walras in the second and following editions of that book. On the changes undergone by this passage from the first to the second edition, see also Walras ([1988], p. 71-72).

brokers with orders to sell. [...] Brokers [...] make bids at 60 francs 05 centimes. They raise the market price. Third supposition. Brokers with orders to sell can no longer find brokers with orders to buy. [...] Brokers [...] make offers at 59 francs 95 centimes. They lower the price".

This passage reveals that the starting point of Walras's analysis is indeed represented by a very realistic picture of the trading process, a picture which apparently stands at a very great distance from the highly stylized image of the same process emerging from the basic assumptions and the formal model presented above. The first striking difference lies in the following: while the model deals with an economy where two commodities proper are traded for one another, the example concerns instead a market where a commodity proper is exchanged for money. Since, as we shall see, Marshall's "temporary equilibrium" model deals precisely with a market where a commodity proper ("corn") is exchanged for money[13], it is particularly important, for our present purposes, to clarify Walras's position in this respect. Now, concerning this point, Walras is fortunately very clear. For, a few lines after the securities example quoted above, he adds: "*Securities, however, are a very special kind of commodity. Furthermore, the use of money in trading has peculiarities of its own, the study of which must be postponed until later, and not interwoven at the outset with the general phenomenon of value in exchange. Let us, therefore, retrace our steps and state our observations in scientific terms. We may take any two commodities, say oats and wheat, or, more abstractly, (A) and (B)*" ([1954], p. 86-87).

In sharply disconnecting the introductory example from the "scientific" discussion of the problem of the exchange of two commodities for one another, Walras takes due care of restoring the symmetry between the two commodities composing the economy under discussion, a symmetry that had been broken, in his example, by the existence, side by side, of such heterogeneous objects as money, with its "peculiarities", and a commodity proper. Precisely, in his "scientific" treatment of the problem at hand, no money exists in any other sense than possibly that of being a unit of account; at the same time, either commodity can indifferently be taken as the numeraire of the economy. As we shall see, this restored symmetry, which sharply distinguishes Walras's formal treatment of the pure-exchange, two-commodity economy from Marshall's, plays a fundamental role in allowing Walras to generalize his approach to more complex economies and models. Coming now to what we have called the three basic assumptions concerning the trading process, one must admit that, at first sight, all three of them are disconfirmed by the securities example. As to Assumption 2 ("Perfect Competition"), one can see that, in that example, there are traders that "make" the price, in the sense that they make price bids, changing them according to the circumstances of the market ("*They raise the market price"*, "*They lower the prices*"), so that traders cannot apparently be viewed as price-takers and the competitive assumption fails. But, since prices are individually changed by traders experiencing rationing, one cannot apparently be sure that different prices will not be quoted by different traders at the same time, so that also Assumption 1, the so-called "*Law of One Price*", would not apply in this case; nor can one exclude the possibility that some transactions be actually carried out at out-of-equilibrium prices, so that Assumption 3, the "*No Trade out of Equilibrium*" assumption, would fail as well.

Now, also with respect to the basic assumptions concerning the trading process, one should be careful in distinguishing a mere illustrative example, which may reasonably be expected to be realistic and captivating, hence also somewhat imprecise, from a formal theoretical model, of which, on the contrary, one should demand absolute rigor and precision. But, in the case at hand, even in the more formal parts of his discussion, Walras's defence of the fundamental assumptions underlying his pure-exchange model is not always so convincing as one might hope

[13] It may be interesting to note that, in Walras' first theoretical work, predating the publication of the first installment of the first edition of the *Eléments* in 1874 and concerning precisely the theory of the exchange of two commodities for one another, one can find an example which is virtually identical with the example in the *Eléments*, except that the commodity proper traded for money in the market under discussion is "corn", instead of being "3 per cent French Rentes"([1874], p. 31-32).

for. Even if the major difficulties concern Assumption 3, we prefer to proceed in order, starting from Assumption 1. The "Law of One Price" is also referred to in the literature as "Jevons' Law of Indifference", since an apparently similar assumption was first introduced into the theoretical debate, under the name of "Law of Indifference", by Jevons in his path-breaking book *The Theory of Political Economy* (1871)[14]. Both Walras's and Marshall's investigations into the problem of exchange are deeply affected by Jevons' "theory of exchange", as developed in Chapter 4 of his 1871 book. In view of this, before going back to Walras's "Law of One Price", it is convenient to briefly discuss the role played by the "Law of Indifference" in Jevons' model of exchange. Jevons' "theory of exchange" is developed with reference to an Edgeworth Box economy, where each one of two "cornered" traders, called by Jevons "trading bodies", owns the total endowment of one of the two commodities existing in the economy. Jevons' problem is to determine the total quantities of commodities 1 and 2 exchanged by the two traders, which are respectively denoted x and y. Then, assuming Jevons' version of the "Law of Indifference" to hold, it turns out that such quantities also implicitly define the equilibrium rate of exchange between the two commodities, y/x, that is, the equilibrium relative price of commodity 1 in terms of commodity 2.

According to Jevons, the "Law of Indifference", "a general law of the utmost importance in economics", can be stated as follows: "*[I]n the same open market, at any one moment, there cannot be two prices for the same kind of article*" ([1871], p. 137). Now, this "Law" allegedly plays a fundamental role in the solution of Jevons' Edgeworth Box model, as can be seen from the following passage: "*Thus, from the self-evident principle* [i.e., the "Law of Indifference"], *stated on p. 137, that there cannot, in the same market, at the same moment, be two different prices for the same uniform commodity, it follows that the last increments in an act of exchange must be exchanged in the same ratio as the whole quantities exchanged. [...] This result we may express by stating that the increments concerned in the process of exchange must obey the equation* $\frac{dy}{dx} = \frac{y}{x}$" ([1871], p. 138-139). While the first italics are Jevons', the second ones are added).

The above equation, which provides one of Jevons' well-known equilibrium conditions, can be interpreted as stating that, "in the process of exchange", the marginal rate of exchange, $\frac{dy}{dx}$, must equal the average rate, $\frac{y}{x}$, which in turn represents the relative price of commodity 1 in terms of commodity 2 in the "act of exchange" concerned. As can be easily verified, however, a serious ambiguity surrounds the interpretation of Jevons' equation: for the equality between marginal and average rate of exchange can only be inferred from the "Law of Indifference" if it is also assumed that the "whole quantities exchanged", x and y, are traded "at the same moment" as "the last increments", dx and dy; this means, however, that "the process of exchange" cannot be distinguished from "an [instantaneous] act of exchange", as the very wording of Jevons' sentence unwillingly reveals. This also implies that the equilibrium rate of exchange be instantaneously reached, so that all transactions can take place at that rate (actually, there will occur just one single grand transaction, that will of course take place at the equilibrium rate). To sum up, contrary to what Jevons appears to suggest, it is not true that the above equation "follows" from the "Law of Indifference" as such: as a matter of fact, it "follows" from the "Law of Indifference" *and* the assumption that the "process of exchange" be a degenerate durationless process. If the "process of exchange" were allowed to be a true time-consuming

[14] What we have called here the "Law of One Price" is also occasionally referred to in the literature as the "*principle of completeness, or universality, of markets*", or else as the assumption of "*universal price quoting of commodities (market completeness)*": see, e.g., Mas-Colell, Whinston, Green ([1995], p. 20, 550). Yet these labels appear to be misnomers and should be avoided in this context.

process, then the "Law of Indifference", as formulated by Jevons, would be perfectly consistent with a sequence of transactions taking place at different instants at different rates of exchange, so that Jevons' equation would not hold true. The uncertainty about the epistemological status of "Jevons' Law of Indifference", which is alternatively interpreted by Jevons himself as either a "self-evident principle", which holds identically true under all circumstances, or as an equilibrium condition, which only holds true under special circumstances, is not entirely dispelled by Walras either. In fact, there are passages where Walras appears to interpret the "Law of One Price" as an equilibrium condition, for example when he states that there can be only one price in the market, namely the price at which total effective demand equals total effective offer [...], or when he summarizes his analysis of the two-commodity, pure-exchange economy by means of the following proposition, which, according to him, "embraces the whole of the pure and applied economics": *"The exchange of two commodities for each other in a perfectly competitive market is an operation by which all holders of either one, or of both, of the two commodities can obtain the greatest possible satisfaction of their wants consistent with the condition that the two commodities are bought and sold at one and the same rate of exchange throughout the market"* ([1954], p. 143).

Yet, the interpretation of the "Law of One Price" as a pure equilibrium condition, though supported by renowned interpreters of Walras's thought (such as Morishima ([1977], p. 11-26)), is ultimately unacceptable. For, as we have seen in the previous subsection, one of the distinguishing features of Walras's model is its relying on the concept of aggregate demand and excess demand function. Now, if it is true that, by providing the market-clearing condition, the nullity of the aggregate excess demand function (equation (3') above) plays a fundamental role in defining the Walrasian equilibrium concept, it is also true that the very notion of aggregate excess demand function could not even be defined if a uniform price, allegedly known to all traders, could not be supposed to exist in any case[15]. Thus, it can be seen that the very structure of Walras's model implies the universal validity of the "Law of One Price", which must be supposed to hold under all circumstances, that is, both at equilibrium and out of equilibrium.

A similar reasoning applies to Assumption 2 : for the way in which Walras constructs the individual demand and excess demand functions, or "trader's schedules", as Walras calls them, leaves no doubt as to the fact that, for the purposes of the theory, he imagines the traders to take commodity prices (a single relative price, in the case at hand) as given parameters and to determine the quantities to be traded of the various commodities (two, in the case under discussion) in such a way as to maximize their utility functions[16]. Hence we can conclude that the "Perfect Competition" assumption, apparently disconfirmed by the securities example, is never really questioned by Walras in his formal model.

In discussing the status of Assumptions 1 and 2 in Walras's model of a pure-exchange, two-commodity economy, we have ascertained that such Assumptions hold in every case, that is, both at equilibrium and out of equilibrium. Up to now, however, we have not yet specified the exact nature of the disequilibrium states that can be regarded as consistent with Walras's overall approach. This is not accidental, for the answer to this question crucially depends on the meaning and implications of Assumption 3, to which we now turn our attention. Assumption 3 is the most problematic of all three: such controversial character is partly due to the fact that, in all probability, Walras did not initially realize the need for such an Assumption. As a matter of fact, Walras's original discussion of this issue - in both his early theoretical writings, such as the 1874 and 1876 *mémoires* on the theory of exchange, and the first edition of the *Eléments* (1874-1877)

[15]On the construction of the aggregate excess demand function see, e.g., Walras ([1954], p. 94-95).

[16] It should be added that, also in discussing the construction of individual demand and excess demand functions, Walras supposes the traders "*to anticipate all possible values of [the price] from zero to infinity and determine accordingly all the corresponding values of [their excess demands]*" (Walras [1954], p. 92); this means that traders are supposed to take all sorts of prices, both equilibrium and disequilibrium ones, as given parameters, behaving competitively under all circumstances. See also, e.g., Walras ([1954], p. 122).

- is highly ambiguous. To be precise, not only the securities example, but also the entire formulation of the pure-exchange model in the 1874 and 1876 *mémoires* and in the first edition of the *Eléments*, are not inconsistent, to say the least, with the idea that some transactions may actually be carried out at disequilibrium prices. Moreover, should we extend our consideration to the production and capital formation models, we would immediately discover that, in the first three editions of the *Eléments* (hence up to 1896), such models explicitly contemplate out-of-equilibrium transactions and other observable disequilibrium activities[17].

But to allow out-of-equilibrium trades to actually occur in the economy is inconsistent with the requirements of equilibrium determination in Walras's approach. To see why, let us focus attention, once again for the sake of simplicity, on the pure-exchange model exclusively. In this model the occurrence of disequilibrium transactions would make the equilibrium indeterminate not only by altering the data of the economy (namely, the individual endowments), but also, and foremost, by changing such data in an unpredictable way: for Walras's theory is indeed able to predict the plans of action optimally chosen by the traders at both equilibrium and disequilibrium prices, but it can only predict the traders' actions (that is, their observable behavior) when the economy is at equilibrium.

These critical remarks, confusedly made by Bertrand in his 1883 review-article of the second edition of Walras's *Théorie mathématique de la richesse sociale* (1883), where Walras's 1874 *mémoire* on the theory of exchange had been reprinted without any significant change, induced Walras to explicitly introduce a "No Trade out of Equilibrium" assumption into his theoretical system, first by dropping a short statement to this effect in an obscure article on Gossen published in 1885, and then, with specific reference to the pure-exchange model, by inserting a few well-chosen words into the securities example in the second (1889) and following editions of the *Eléments*: precisely, in discussing the three alternative "suppositions" which are separately analyzed in that example, "according as the demand is *equal to*, *greater than*, or *less than* the offer", Walras added the statement "Exchange takes place" in the case of market equilibrium, while he inserted the short sentences "Theoretically, trading should come to a halt" and "Trading stops" in the cases of excess demand and excess supply, respectively ([1954], p. 85). As to the production and capital formation models discussed in the various editions of the *Eléments*, however, a sort of "No Trade out of Equilibrium" assumption was only introduced in the fourth edition of the *Eléments*, published in 1900, when Walras eventually resolved to adopt the so-called "hypothèse des *bons*": according to this assumption, all traders (that is, not only consumers, as in the pure-exchange model of the second and subsequent editions of the *Eléments*, but also producers and owners of the factors of production) are not allowed to carry out any actual transactions until an equilibrium is arrived at; until then, they can only exchange "*bons*", that is, conditional claims, which are not effective whenever the economy is out of equilibrium ([1954], p. 242, 282, 319). So, Assumption 3 is eventually vindicated, becoming one of the cornerstones of the Walrasian edifice in its final form. But it would be misleading to conceal that it took more than a quarter of a century to Walras to convince himself that his theory cannot not do without such an assumption.

4. Walras's model : limitations and extensions

The reason why Walras so strenuously resisted the generalized adoption of the "No Trade out of Equilibrium" assumption is easy to explain. This assumption, when combined with the other two, turns the process of adjustment towards equilibrium into a purely virtual process, where nothing observable can occur. Such virtual process evolves over a "logical" time entirely disconnected from the "real" time over which the economy is supposed to evolve. Hence, since it takes just one instant of "real" time for the adjustment process to carry its effects through, the equilibrium state, granting that it is eventually reached, must be imagined as "instantaneously" arrived at, as

[17]A thorough discussion of the evolution of Walras' ideas concerning equilibrium, disequilibrium and the equilibration process, the celebrated *tâtonnement* process, can be found in Donzelli (2005).

far as the "real" time of the economy is concerned. But this "instantaneous" character of the equilibrium concept, which Walras is eventually, though unwillingly, led to recognize[18], clashes with his original idea that the empirical content of general equilibrium theory crucially depends on the possibility of showing that an equilibrium state "comes to be established" through an adjustment process in "real" time, where observable behavior is allowed both to take place and to play an essential role out of equilibrium.

So, it is true that Walras's basic assumptions about the nature of trading process severely restrict the claims that his equilibrium approach, and especially the underlying theory of the equilibration process, can lay to descriptive realism. And it is also true that such restrictions are difficult to swallow, first of all for Walras himself, as the length of the period needed to accept them witnesses. But in the end he is willing to take this step, because he is aware that accepting those constraints is the price to be paid for achieving not only a theoretical consistency, but also a descriptive generality, that would be unattainable otherwise.

In fact, this can be easily seen by going back to the pure-exchange, two-commodity model from which we started, and analyzing the role played by Walras's various assumptions in allowing him to extend the scope of this simple model, in such a way as to progressively encompass, in a very natural way, an ever larger set of economic issues and phenomena. In the first place, it should be stressed that, by assuming from the very beginning the "Law of One Price" and "Perfect Competition", Walras can straightforwardly attack the problem of equilibrium determination in an economy with *any* finite number of traders, without being forced either to confine his analysis to a two-trader economy, as Jevons (1871) had been forced to do, or to make further special assumptions on the traders' characteristics, as Marshall will be compelled to do in his *Principles* (1890), as we shall see in the next section. Moreover, with regard to the traders' characteristics, it should also be added that Walras's original assumptions concerning the cardinality and additive separability of the traders' utility functions turn out to be unnecessarily restrictive, even if Walras will never become aware of this, and can be easily disposed of, as Pareto (1906) will prove a few years later, without jeopardizing in the least Walras's approach and results in dealing with the pure-exchange problem with any number of traders. Furthermore, by temporarily giving up the apparently realistic pretence to cope with both the exchange and the money issue at one and the same time within the pure-exchange, two-commodity model, and by choosing from the beginning to normalize the price system by taking one commodity proper, instead of money, as the numeraire of the economy, Walras makes it easier to smoothly generalize his analysis of the two-commodity economy to a multi-commodity world, in a truly general equilibrium framework[19]. Finally, by complementing the "Law of One Price" and the "Perfect Competition" assumption with the "No Trade out of Equilibrium" assumption, Walras makes it possible to apply the same analytical apparatus and the same "instantaneous" equilibrium concept, already employed with reference to pure-exchange economies with an arbitrary number of traders, to more general economies with production, capital formation, and even money, which can eventually be reintroduced into the analysis. As can be seen, therefore, a sort of trade-off between realism, on the one hand, and consistency and generality, on the other, seems to apply in Walras's case: giving up a relatively more realistic analysis of the

[18]As far as the pure exchange model is concerned, Walras recognizes the "instantaneous" character of his equilibrium construct as early as in 1885, in the already quoted article on Gossen ([1885], p. 312, fn. 1). Instead, as far as the more comprehensive models with production, capital formation, circulation and money are concerned, one has to wait for the well-known passage of Lesson 29, newly added to the fourth edition of the *Eléments* (1900), where the implications of the so-called "hypothèse des *bons*" for the time structure of the analysis and the nature of the equilibrium construct are extensively discussed ([1954], p. 319).

[19]This extension is carried out by Walras himself in Lesson 11 of the *Eléments*, which is the first Lesson of Part III of that book, entitled "Theory of Exchange of Several Commodities for One Another". It may be interesting to note that Walras exclusively employs the expression "general equilibrium", later used in a much more comprehensive sense, to denote a state of a multi-commodity, moneyless economy in which a consistent price system, normalized by choosing an appropriate numeraire, obtains.

disequilibrium process appears to be the price to be paid for gaining a sounder consistency and a greater generality in the field of equilibrium theory.

III. Marshall's approach

Let us turn now to Marshall's approach. As explained in section 2, we are essentially concerned here with Marshall's model of an Edgeworth Box economy, as expounded in "Appendix F. Barter" in the fifth and following editions of the *Principles*, as well as with his "market-day" or "temporary equilibrium" model, as developed in Chapter II of Book V of the same treatise. The relationship between Marshall's "temporary equilibrium" model and his more elaborate "normal equilibrium" models will be briefly discussed in subsection 4.4 below. In Marshall's case, for the reasons already stated, we shall try to reconstruct his formal models from a hermeneutical analysis of the available textual evidence, jointly developing theory and interpretation.

1. Marshall's basic assumptions about the trading process

Unlike Walras, Marshall does *not* assume the traders to behave "competitively", if by this expression one means that the traders take prices as given and choose quantities (i.e., choose consumption or trade plans) in such a way as to maximize utility. This means that in Marshall one does *not* find demand or excess demand functions comparable to Walras's, since the latter's functions, as we have seen, essentially depend on the assumption of "Perfect Competition" and the "Law of One Price" (in the sense specified above). What we do find in Marshall is a different kind of functions, which are still related to the idea that the traders behave "rationally" and "competitively", even if Marshall's conception of rationality and competition is different from Walras's[20]. Marshall's fundamental ideas about the trading process are the following: 1) the trading process in a pure-exchange, two-commodity economy should be viewed as a sequence of bilateral bargains, each involving two traders at a time; 2) the conditions governing each individual bargain (quantities traded of the two commodities, hence rate of exchange between them) should be specified by exploiting the general properties of the marginal rate of substitution of one commodity for the other for the two traders involved in the bargain, where the marginal rate of substitution is viewed as the reservation price of either a buyer or a seller, as the case may be.

To develop Marshall's model, let us focus on consumer i. At the beginning of the trading process, let $MRS^i_{21}(\omega_{1i}, \omega_{2i}) = \frac{\partial u_i(\omega_{1i}, \omega_{2i})}{\partial x_{1i}} / \frac{\partial u_i(\omega_{1i}, \omega_{2i})}{\partial x_{2i}} = v'_{1i}(\omega_{1i}) / v'_{2i}(\omega_{2i})$ be the initial value of consumer i's marginal rate of substitution of commodity 2 for commodity 1. Supposing that there exists a consumer $j \neq i$, such that $MRS^j_{21}(\omega_{1j}, \omega_{2j}) \neq MRS^i_{21}(\omega_{1i}, \omega_{2i})$, let

$$k_{ij}(\omega) = \min\{MRS^i_{21}(\omega_{1i}, \omega_{2i}), MRS^j_{21}(\omega_{1j}, \omega_{2j})\}$$

and

$$K_{ij}(\omega) = \max\{MRS^i_{21}(\omega_{1i}, \omega_{2i}), MRS^j_{21}(\omega_{1j}, \omega_{2j})\}.$$

Then a bilateral bargain involving a marginal trade $(dx_{1i}, dx_{2i}) = -(dx_{1j}, dx_{2j})$ between the two consumers is weakly advantageous to both (that is, it increases the utility of at least one of them, without decreasing the utility of the other), as long as the marginal rate of exchange between the two commodities, $\left|\frac{dx_{2i}}{dx_{1i}}\right| = \left|\frac{dx_{2j}}{dx_1 j}\right|$, belongs to the interval $\lfloor k_{ij}(\omega), K_{ij}(\omega)\rfloor$.

[20]In view of this, it is wholly inappropriate and misleading to call "Marshallian", as many well-known advanced microeconomic text-books do (Varian [1992], p. 105-109)), what is to all purposes an ordinary Walrasian demand function, obtained under the standard Walrasian assumptions about individual rationality and market competition, as stated in the previous section.

Now, if one assumes that any weakly advantageous bargain will be exploited by the party (or parties) benefiting from it, one can predict that consumer i's initial allocation will change whenever there exists another consumer who, at his initial allocation, is characterized by a marginal rate of substitution different from i's. But this prediction is obviously insufficient to make the analysis of the trading process involving consumer i determined: to this end, in fact, it would be necessary to know exactly who are the consumers with whom consumer i makes dealings, what is the time order of these dealings, what are the amounts traded in each case, and so on. For the same reasons, even if one can predict that the trading process will eventually come to an end when the marginal rate of substitution is the same for all consumers, for in that case no weakly advantageous bargain is left to be exploited by anybody, at this stage of the analysis, failing further assumptions, one can predict neither the final allocation nor, as a consequence, the final rate of exchange of the two commodities for one another.

2. Marshall's model of an Edgeworth Box economy

According to Marshall, this sort of indeterminacy is intrinsic to any trading process involving two commodities proper, that is, to any "*system of barter*" ([1961a], p. 334). Such kind of trading processes is examined in greater detail in Appendix F of the *Principles* which, as already mentioned, is specifically devoted to the analysis of a "system of barter". To begin with, Marshall makes the simplifying assumption that only two traders be involved in the barter process, thereby turning the economy under question into an Edgeworth Box economy, $\mathcal{E}_{EB} = \left\{ \left(\Re^2_+, u_i(\cdot), \omega_i \right)_{i=1}^{2} \right\}$. The traders' characteristics satisfy all the assumptions made in section 2, with the further restriction, introduced here for reasons similar to those already explained in discussing Walras's model, that the second-order pure partial derivatives of the traders' utility functions are taken to be strictly negative. Under these conditions, Marshall shows, by means of numerical examples, that the barter process between two consumers trading "apples" for "nuts" may follow a number of alternative paths, each of which eventually terminates, "*because any terms that the one is willing to propose would be disadvantageous to the other. Up to this point exchange has increased the satisfaction on both sides, but it can do so no further. Equilibrium has been attained; but really it is not the equilibrium, it is an accidental equilibrium*" ([1961a], p. 791). Specifically, Marshall examines three alternative paths. The first one, characterized by a constant rate of exchange between the two commodities over the exchange process, stands apart from all the other possible paths, occupying a position which, according to Marshall, is theoretically unique, though practically irrelevant: "*There is, however, one equilibrium rate of exchange which has some sort of right to be called the true equilibrium rate, because if once hit upon would be adhered to throughout. [...] This is then the true position of equilibrium; but there is no reason to suppose that it will be reached in practice*" ([1961a], p. 791)[21].

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[21] Marshall provides two apparently similar, but really quite different, definitions of what might called a "true equilibrium rate" or a "true equilibrium price": the first is put forward in the passage of Appendix F to which this footnote is appended; the second, instead, is suggested in a passage appearing in Chapter II of Book V of the *Principles* ([1961a], p. 333), a passage to which we shall come back in the next subsection : "According to the first definition, as we have seen, a certain "equilibrium rate of exchange [...] has some sort of right to be called the true equilibrium rate, because *if once hit upon it would be adhered to throughout*" (italics added). According to the second, instead, a certain "price [...] has [...] some claim to be called the true equilibrium price [...] because *if it were fixed on at the beginning, and adhered to throughout, it would exactly equate demand and supply*" (italics added)". As can be seen, the two definitions share in common the idea that, in order to qualify as a "true equilibrium rate of exchange" (resp., "price"), a "rate of exchange" (resp., "price") should be constant throughout the trading process. But while the first definition seems to require, on top of this, that any such "true equilibrium rate", once accidentally "hit upon", should be deliberately preserved by the traders, the second does not make any such additional request. As will be seen in a moment, however, nothing in Marshall's theory authorizes one to suppose that, throughout the trading process, the traders have any reason to stick to any rate or price upon which they have accidentally stumbled at the beginning or, for that matter, at any stage of the process. Hence Marshall's first definition actually presupposes more than what is justified by his own theory; for this reason, it ought to be discarded in favor of the second

Either one of the other two paths worked out in detail by Marshall is instead characterized by a variable rate of exchange between the two commodities over the trading process: such rate, in fact, is supposed to be monotonically increasing in one case, decreasing in the other. Referring to the latter two cases, deemed to be in some sense representative of a general pattern, Marshall concludes: "*In both these cases the exchange would have increased the satisfaction of both as far as it went; and when it ceased, no further exchange would have been possible which would not have diminished the satisfaction of at least one of them. In each case an equilibrium rate would have been reached; but it would be an arbitrary equilibrium*"([1961a], p. 792).

This discussion can be formalized as follows. Let $i = 1,2$. Assuming $MRS_{21}^{1}(\omega_{11}, \omega_{21}) \neq MRS_{21}^{2}(\omega_{12}, \omega_{22})$, let

$$k_{12}(\omega) = \min\{MRS_{21}^{1}(\omega_{11}, \omega_{21}), MRS_{21}^{2}(\omega_{12}, \omega_{22})\} <$$

$$< \max\{MRS_{21}^{1}(\omega_{11}, \omega_{21}), MRS_{21}^{2}(\omega_{12}, \omega_{22})\} = K_{12}(\omega).$$

Then the Pareto set of $\square\varepsilon_{EB}$ is the set $P_{EB} = \{x^{P} \in A_{pe}^{2\times 2} \mid MRS_{21}^{1}(x_{1}^{P}) = MRS_{21}^{2}(x_{2}^{P})\}$, while the contract curve of ε_{EB} is the set $C_{EB} = \{x^{C} \in P_{EB} \mid u_{1}(x_{1}^{C}) \square u_{1}(\omega_{1}), u_{2}(x_{2}^{C}) \square u_{2}(\omega_{2})\}$. Under the stated assumptions, $C_{EB} \neq \emptyset$.

For Marshall, any allocation $x^{C} \in C_{EB}$ may represent an "equilibrium", and any corresponding common marginal rate of substitution between the two commodities, $MRS_{21}^{i}(x_{i}^{C}) = p_{1}^{C}$, for $i = 1,2$, may represent an "equilibrium rate of exchange" between the commodities concerned. But, in general, any such allocation (resp., rate) would be an "arbitrary" or "accidental" equilibrium allocation (resp., rate). According to Marshall, only a rate of exchange $p_{1}^{*} = MRS_{21}^{1}(x_{1}^{*}) = MRS_{21}^{2}(x_{2}^{*})$ satisfying the additional condition

$$p_{1}^{*} = \left| \frac{x_{21}^{*} - \omega_{21}}{x_{11}^{*} - \omega_{11}} \right| = \left| \frac{x_{22}^{*} - \omega_{22}}{x_{12}^{*} - \omega_{12}} \right|$$

would qualify as a "true equilibrium rate"[22]. In the above quoted passage Marshall seems to imply that there exists exactly one such rate. Yet, while the stated conditions are sufficient for a "true equilibrium rate" to exist in $\square\varepsilon_{EB}$, they are not sufficient for uniqueness: in this respect, therefore, Marshall appears to be overoptimistic. Finally, since $MRS_{21}^{i}(x_{i}^{*}) \equiv \left| \frac{dx_{2i}^{*}}{dx_{1i}^{*}} \right|_{u(x_{i}^{*}+dx_{i}^{*})=u(x_{i}^{*})} = \frac{v_{1i}^{'}(x_{1i}^{*})}{v_{2i}^{'}(x_{2i}^{*})}$, $i = 1,2$, in Marshall's "true equilibrium" the following condition also holds:

$$\frac{v_{1i}^{'}(x_{1i}^{*})}{v_{2i}^{'}(x_{2i}^{*})} = \left| \frac{dx_{2i}^{*}}{dx_{1i}^{*}} \right| = \left| \frac{x_{2i}^{*} - \omega_{2i}}{x_{1i}^{*} - \omega_{1i}} \right|, i = 1,2,$$

definition, as we shall do in the following.

[22]We stick here to Marshall's second definition of a "true equilibrium rate", which simply requires the rate of exchange to be constant throughout the trading process, without implying that the traders have any reason whatsoever to adhere to it.

which is nothing but Jevons' well-known equilibrium condition ([1871], p. 142-143)[23]. These conclusions would not change if the economy consisted of any larger, but finite, number of traders[24]. For, according to Marshall, the indeterminacy of the final (or equilibrium) rate of exchange between the two commodities, equal to the marginal rate of substitution common to all traders in the final allocation, does not essentially depend on the number of traders in the economy. Rather, "*[the] uncertainty of the rate at which the equilibrium is reached depends indirectly on the fact that one commodity is being bartered for another instead of being sold for money. For, since money is a general purchasing medium, there are likely to be many dealers who can conveniently take in, or give out, large supplies of it; and this tends to steady the market*" ([1961a], p. 793).

As far as the indeterminacy problem is concerned, the fundamental property of money, which is not generally shared by commodities proper, is that, owing to its large supply and general diffusion among the traders, "*its marginal utility is practically constant*"[25]. In Marshall's terminology, the theory dealing with those trading processes in which one side of each bargain is in the form of "money", the other being in the form of a commodity proper, is called the "theory of buying and selling". Towards the end of Appendix F of the *Principles*, Marshall contrasts the "theory of buying and selling" with the "theory of barter", stressing what he regards as the essential difference between the two: " *The real distinction then between the theory of buying and selling and that of barter is that in the former it generally is, and in the latter it generally is not, right to assume that the stock of one of the things which is in the market and ready to be exchanged for the other is very large and in many hands; and that therefore its marginal utility is practically constant*"([1961a], p. 793). In view of this, going back to the Edgeworth Box example already discussed in the first part of the Appendix, but assuming now that one of the two commodities traded ("nuts") shares the essential properties of money (large supply and general diffusion, hence "constant marginal utility"), while the other ("apples") does not, Marshall categorically asserts that, independently of the path followed by the exchange process, "[i]n this case the bargaining *must* issue in [a determinate outcome]": precisely, what turns out to be determined in this case is both the total quantity traded of the commodity proper ("apples") and the final rate of exchange between the two commodities. The latter, being uniquely determined, can legitimately be said in this case to represent "*the* equilibrium" rate, rather than simply "*an*" (or "an accidental" or " an arbitrary") "equilibrium" rate; but it might also be legitimately qualified as "*the true equilibrium rate*", because it does satisfy the condition set out by Marshall ([1961a], p. 333) for so qualifying a rate of exchange. What instead remains undetermined, even in this special case, is the total quantity traded of the money-like commodity ("nuts"), which depends on the specific path followed by the trading process[26].

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[23]As can be seen, Marshall interprets the equality between marginal and average rate of exchange as an equilibrium condition: precisely, he interprets it as the defining condition of a "true equilibrium". As has been shown in subsection 3.3 above, a similar line of reasoning had already been suggested by Jevons (1871), so that it is by no means surprising that Marshall's condition for a "true equilibrium" should coincide with Jevons' original condition for an equilibrium of his model. In spite of these obvious similarities, however, there are two important differences between Marshall's approach and that adopted many years before by Jevons. First, while Jevons tries (without success) to justify the assumed equality of marginal and average rate of exchange as a necessary consequence of his "Law of Indifference", Marshall does not endorse such contrived appeal to "Jevons' Law" (as we shall see later, Marshall is only occasionally willing to accept a milder version of the "Law of Indifference"). Second, while Jevons uses the equality between marginal and average rate of exchange to actually solve his model, Marshall does not make any effective use of such equality, apart from defining what he calls the "true equilibrium" concept; but, as we have already seen, the "true equilibrium" concept is practically irrelevant for Marshall, who in effect follows a completely different route to find the solution of his model.

[24] Marshall ([1961a], p. 792). Here Marshall, without explicitly mentioning Edgeworth, is clearly attacking the latter's theory of recontracting, as put forward in Edgeworth (1881).

[25]As we shall see in the next subsection, another condition is in effect required, according to Marshall, for the marginal utility of money to be approximately constant in real-world trading processes.

[26] ([1961a], p. 791, 793; Marshall's italics). The issue of equilibrium determinacy in Marshall's theory of barter was critically discussed by Edgeworth in an article in Italian, published on an Italian journal one year after the

Let us now verify whether the results allegedly reached by Marshall in the framework of his particular example actually hold in the formal model of a special Edgeworth Box economy, $\varepsilon_{EB}^{m} = \varepsilon_{pe}^{2\times 2,m}$, where commodity 1 ("apples") is a commodity proper, while commodity 2 ("nuts") is a money-like commodity, whose marginal utility is assumed to be constant (the superscript m in both ε_{EB}^{m} and $\varepsilon_{pe}^{2\times 2,m}$ is there to remind the reader of the money-like nature of one of the two commodities). In view of the money-like character ascribed to commodity 2, it is natural to take that commodity as the numeraire in this model, so that $p_2 \equiv 1$. As to commodity 1, we shall see that many different concepts of the price of commodity 1 in terms of commodity 2 need to be employed in order to formalize Marshall's approach: namely, for each consumer i, both a "demand price" and a "supply price" of commodity 1 in terms of commodity 2 will be defined in the following; moreover, an "equilibrium price" concept will be needed as well (but the latter, as already seen, may require further qualifications, for it may be either "true" or "arbitrary" and "accidental", as the case may be). In any case, the price of commodity 1 in terms of commodity 2 will always be denoted by p_1 in what follows, with additional subscripts or superscripts specifying the particular price concept at issue.

Marshall's "constant marginal utility of money" assumption can be formally rendered by assuming consumer i's utility function to be quasi-linear in commodity 2, that is:

$$u_i(x_{1i}, x_{2i}) = v_{1i}(x_{1i}) + x_{2i}, \quad i = 1,2,$$

where the constant marginal utility of the money-like commodity has been normalized to 1, i.e., $\partial u_i(x_{1i}, x_{2i})/\partial x_{2i} = 1 = constant$, so that in this case one also has $\partial^2 u_i(x_{1i}, x_{2i})/\partial x_{2i}^2 = 0$. As we have seen, Marshall's main empirical justification for adopting the "constant marginal utility assumption" is that money is in large and general supply. It is difficult to formalize this empirical condition in an Edgeworth Box economy. In any case, we shall assume that consumer i's endowment of the money-like commodity is "sufficiently large", that is, ω_{2i} is no less than a positive number $m_i > 0$, to be specified in due time, for $i = 1,2$. Marshall's assumptions concerning the marginal utility function of a commodity proper can be rendered by means of the following restrictions on the partial derivatives of consumer i's utility function with respect to the quantity consumed of commodity 1:

$$\frac{\partial u_i(x_{1i}, x_{2i})}{\partial x_{1i}} = v_{1i}'(x_{1i}) > 0, \; \frac{\partial^2 u_i(x_{1i}, x_{2i})}{\partial x_{1i}^2} = v_{1i}''(x_{1i}) < 0,$$

for $x_{1i} \geq 0$, $i = 1,2$ [27]. Under these assumptions we have $MRS_{21}^{i}(x_i) = \dfrac{\partial u_i(x_i)}{\partial x_{1i}} / \dfrac{\partial u_i(x_i)}{\partial x_{2i}} = v_{1i}'(x_{1i})$, so that the marginal rate of substitution of commodity 2 for commodity 1, or the reservation price of commodity 1 in terms of commodity 2, only depends on the quantity consumed of commodity 1. The latter, as we shall see, is the property of the traders' characteristics driving Marshall's results in his Edgeworth Box model: for this reason it will be referred to as "Marshall's fundamental property" in the sequel[28].

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appearance of the first edition of the *Principles* (Edgeworth, 1891a). Edgeworth's criticism was rebutted by a Cambridge mathematician, Arthur Berry (1891), who published his reply to Edgeworth on the same journal at Marshall's instigation. Edgeworth's rejoinder (1891b) ended the controversy. On this controversy see also Marshall's comments in Note XII *bis* of the Mathematical Appendix of the *Principles* ([1961a], p. 844-845), as well as the editorial notes and the letters to Edgeworth by Marshall and Berry, respectively, in ([1961b], p. 790-798). See also Newman's notable contribution in Whitaker (1990).

[27]([1961a], p. 93 and 838). Also in Marshall's case, as we already did in Walras' case and essentially for the same reasons, we shall exclude the possibility of satiation, even if Marshall does not rule it out ([1961a], p. 93, fn. 1). This strong monotonicity assumption, however, can be dispensed with, at the cost of complicating somewhat the analysis.

[28]Since the marginal rate of substitution is invariant under any arbitrary strictly increasing transformation of the utility index, all properties of the marginal rate of substitution, including its independence of the amount of the money-like commodity in the consumption bundle, can be regarded as ordinal properties.

Let $d_{1i}(x_{1i}, \omega_{1i}) = \max\{0, x_{1i} - \omega_{1i}\}$ be consumer i's net demand proper for commodity 1 and $s_{1i}(x_{1i}, \omega_{1i}) = |\min\{0, x_{1i} - \omega_{1i}\}|$ be his net supply of commodity 1, for $x_{1i} \geq 0$, $i = 1,2$. If $x_{1i} > \omega_{1i}$, then $d_{1i}(x_{1i}, \omega_{1i}) > 0$ and consumer i is a net buyer of commodity 1; hence $MRS_{21}^{i}(x_{1i}) = v_{1i}^{'}(x_{1i})$ can be interpreted as a buyer's reservation price, or demand price, that is as the maximum quantity of commodity 2 that consumer i is willing to pay in exchange for one unit of commodity 1, when his present consumption of commodity 1 is x_{1i}. If $x_{1i} < \omega_{1i}$, then $s_{1i}(x_{1i}, \omega_{1i}) > 0$ and consumer i is a net seller of commodity 1; hence $MRS_{21}^{i}(x_{1i}) = v_{1i}^{'}(x_{1i})$ can be interpreted as a seller's reservation price, or supply price, that is as the minimum quantity of commodity 2 that consumer i is willing to receive in exchange for one unit of commodity 1, when his present consumption of commodity 1 is x_{1i}. Finally, if $x_{1i} = \omega_{1i}$, then $d_{1i}(\omega_{1i}, \omega_{1i}) = s_{1i}(\omega_{1i}, \omega_{1i}) = 0$ and consumer i is neither a net buyer nor a net seller of commodity 1, so that $MRS_{21}^{i}(\omega_{1i}) = v_{1i}^{'}(\omega_{1i})$ can be interpreted as both the maximum quantity of commodity 2 that consumer i is willing to pay and the minimum quantity of commodity 2 that consumer i is willing to receive in exchange for one unit of commodity 1, when his present consumption of commodity 1 is ω_{1i}.

Hence, given $s_{1i}(x_{1i}, \omega_{1i}) \in (0, \omega_{1i})$, let $p_{1i}^{s}(s_{1i}(x_{1i}, \omega_{1i})) = v_{1i}^{'}(\omega_{1i} - s_{1i}(x_{1i}, \omega_{1i})) = v_{1i}^{'}(x_{1i})$ be consumer i's supply price of commodity 1 when his consumption of that commodity is $x_{1i} = \omega_{1i} - s_{1i}(x_{1i}, \omega_{1i})$;similarly, given $d_{1i}(x_{1i}, \omega_{1i}) \geq 0$, let $p_{1i}^{d}(d_{1i}(x_{1i}, \omega_{1i})) = v_{1i}^{'}(\omega_{1i} + d_{1i}(x_{1i}, \omega_{1i})) = v_{1i}^{'}(x_{1i})$ be consumer i's demand price of commodity 1 when his consumption of that commodity is $x_{1i} = \omega_{1i} + d_{1i}(x_{1i}, \omega_{1i})$. The correspondence $p_{1i}^{s} : [0, \omega_{1i}] \rightarrow \Re_{+}$, mapping consumer i's net supplies of commodity 1 into consumer i's supply prices of commodity 1, is called consumer i's Marshallian inverse supply correspondence of commodity 1. The correspondence $p_{1i}^{s}(\cdot)$ is defined as follows: $p_{1i}^{s}(s_{1i}) = [0, v_{1i}^{'}(\omega_{1i})]$, for $s_{1i} = 0$; $p_{1i}^{s}(s_{1i}) = v_{1i}^{'}(\omega_{1i} - s_{1i})$, for $s_{1i} \in (0, \omega_{1i})$; $p_{1i}^{s}(s_{1i}) = [v_{1i}^{'}(0), \infty)$ for $s_{1i} = \omega_{1i}$. Given the assumptions on $v_{1i}(\cdot)$ and its derivatives, the restriction of $p_{1i}^{s}(\cdot)$ to the domain $(0, \omega_{1i})$ is a strictly increasing continuous function. Similarly, the correspondence $p_{1i}^{d} : [0, \infty) \rightarrow \Re_{+}$, mapping consumer i's net demands for commodity 1 into consumer i's demand prices of commodity 1, is called consumer i's Marshallian inverse demand correspondence for commodity 1. The correspondence $p_{1i}^{d}(\cdot)$ is defined as follows: $p_{1i}^{d}(d_{1i}) = [v_{1i}^{'}(\omega_{1i}), \infty)$, for $d_{1i} = 0$; $p_{1i}^{d}(d_{1i}) = v_{1i}^{'}(\omega_{1i} + d_{1i})$, for $d_{1i} > 0$. Given the assumptions on $v_{i}(\cdot)$ and its derivatives, the restriction of $p_{1i}^{d}(\cdot)$ to the domain $(0, \infty)$ is a strictly decreasing continuous function.

This has prompted Newman ([1990], p. 265) to suggest that Marshall's cardinal interpretation of the traders' utility functions, and specifically his quasi-linearity assumption (that is, the assumption that the utility functions be additively separable in the amounts of the two commodities and linear in the second one, which in turn implies the constancy of the marginal utility of the money-like commodity), though sufficient for Marshall's main result, are not necessary for it and can be dispensed with at no cost. This conclusion, however, is questionable, not only on general methodological grounds, as explained by Mas-Colell, Whinston and Green (1995, p. 50) in their discussion of cardinality and quasi-linearity, but also with specific reference to Marshall's problem, should one attempt - as Newman does ([1990], p. 265-267) - to generalize Marshall's approach from a two-commodity economy with a money-like commodity to a multi-commodity economy with money. In fact, when there is more than one commodity proper in the economy, "Marshall's fundamental property" can only be preserved by assuming the traders' utility functions to be additively separable in all their arguments (i.e., amounts of commodities proper and money) and quasi-linear in money.

By first taking the inverses of the previous two functions, and then suitably extending such inverses to cover the whole price domain, one gets the Marshallian direct supply and demand functions. Namely, consumer i's Marshallian direct supply function of commodity 1, mapping consumer i's supply prices into net supplies of commodity 1, is the continuous function $s_{1i} : \Re_+ \to [0, \omega_{1i}]$ defined as follows: $s_{1i}(p_{1i}^s) = 0$, for $p_{1i}^s \in [0, v_{1i}^{'}(\omega_{1i}))$; $s_{1i}(p_{1i}^s) = \omega_{1i} - (v_{1i}^{'})^{-1}(p_{1i}^s)$, for $p_{1i}^s \in [v_{1i}^{'}(\omega_{1i}), v_{1i}^{'}(0))$; $s_{1i}(p_{1i}^s) = \omega_{1i}$, for $p_{1i}^s \in [v_{1i}^{'}(0), \infty)$. The function $s_{1i}(\cdot)$ is nondecreasing in p_{1i}^s, and strictly increasing for $p_{1i}^s \in [v_{1i}^{'}(\omega_{1i}), v_{1i}^{'}(0)]$. Similarly, consumer i's Marshallian direct demand function for commodity 1, mapping consumer i's demand prices into net demands of commodity 1, is the continuous function $d_{1i} : \Re_{++} \to [0, \infty)$ defined as follows: $d_{1i}(p_{1i}^d) = 0$, for $p_{1i}^d \in [v_{1i}^{'}(\omega_{1i}), \infty)$; $d_{1i}(p_{1i}^d) = (v_{1i}^{'})^{-1}(p_{1i}^d) - \omega_{1i}$, for $p_{1i}^d \in (0, v_{1i}^{'}(\omega_{1i}))$. The function $d_{1i}(\cdot)$ is nonincreasing in p_{1i}^d, and strictly decreasing for $p_{1i}^d \in (0, v_{1i}^{'}(\omega_{1i})]$.

Now, assuming consumer i's preferences to be such that i's potential expenditure on commodity 1 be bounded above, the restriction on consumer i's minimum endowment of the money-like commodity can be specified as follows:

$$\omega_{2i} \geq \bar{m}_i = \sup_{d_{1i} \in [0, \infty)} \left\{ p_{1i}^d (d_{1i}) d_{1i} \right\}, \; i = 1,2.$$

Now let $d_1(p_1^d) = \sum_{i=1}^{2} d_{1i}(p_{1i}^d)$, for $p_1^d = p_{1i}^d$, $i = 1,2$, and $p_1^d \in (0, \infty)$, and let $s_1(p_1^s) = \sum_{i=1}^{2} s_{1i}(p_{1i}^s)$, for $p_1^s = p_{1i}^s$, $i = 1,2$, and $p_1^s \in [0, \infty)$. The functions $d_1(\cdot)$ and $s_1(\cdot)$, arrived at by aggregating the individual demand and supply functions over all consumers, are called the Marshallian aggregate demand and supply functions for commodity 1, respectively. Let $p_{1\max}^d = \max_i \{v_{1i}^{'}(\omega_{1i})\}$, $p_{1\min}^s = \min_i \{v_{1i}^{'}(\omega_{1i})\}$ and $p_{1\max}^s = \max_i \{v_{1i}^{'}(0)\}$, $i = 1,2$. Then the function $d_1(\cdot)$ is nonincreasing in p_1^d, and strictly decreasing for $p_1^d \in (0, p_{1\max}^d]$, while the function $s_1(\cdot)$ is nondecreasing in p_1^s, and strictly increasing for $p_1^s \in [p_{1\min}^s, p_{1\max}^s]$. Further, provided that consumers' preferences be not identical at the initial allocation, $p_{1\max}^d > p_{1\min}^s$. Hence, there must exist a unique price $p_1^M = p_1^{dM} = p_1^{sM} \in (p_{1\min}^s, p_{1\max}^d)$ such that

$$d_1(p_1^M) = s_1(p_1^M) \qquad (4')$$

or

$$d_1(p_1^M) - s_1(p_1^M) = 0 \qquad (4'')$$

where p_1^M may be called the Marshallian equilibrium price of commodity 1 in terms of commodity 2, while the common value $d_1(p_1^M) = s_1(p_1^M)$, synthetically denoted by $q_1(p_1^M)$, may be called the Marshallian equilibrium total traded quantity of commodity 1 or, for short, the equilibrium quantity of commodity 1. Equation (4”) closely resembles the Walrasian equilibrium equation (2’), embodying the market-clearing condition for commodity 1, any solution of which represents a Walrasian equilibrium price of commodity 1 in terms of commodity 2, p_{12}^W. But, all similarities notwithstanding, the interpretation of equation (4”), and specifically of the associated Marshallian equilibrium price concept, is altogether different from that of equation (2’), and specifically of the associated Walrasian equilibrium price concept. As a matter of fact, in spite of its appearance, equation (4”) (or, for that matter, equation (4’)) is *not* a market-clearing equation; similarly, in spite of its apparent role, p_1^M is *not* a market-clearing price. It is certainly true that, if the two consumers should agree to carry out all their trades at a constant rate of exchange equal to p_1^M, then the market for commodity 1 would "clear" at that rate, in the sense that, at the end of the trading process, the total quantity traded of commodity 1 would be equal to both the quantity demanded and the quantity supplied, that is, to the common value $q_1(p_1^M) = d_1(p_1^M) = s_1(p_1^M)$. But typically the two consumers will not carry out their trades at the constant rate p_1^M; and yet, even if different trades take place at different rates, at the end of the process the total quantity traded of commodity 1 will still be equal to the common value $q_1(p_1^M)$. But then, if the rate p_1^M does not play any exclusive market-clearing role, since the market "clears", in the sense specified, also with a non-constant sequence of rates of exchange, what is exactly the role played by p_1^M? And why does the Marshallian equilibrium total traded quantity of commodity 1 invariably equal $q_1(p_1^M)$?

When the two consumers have already cumulatively traded a quantity $\hat{q}_1$ of commodity 1, such that $\hat{q}_1 \in [0, q_1(p_1^M))$, there still exists a positive difference between the demand and the supply price of commodity 1 corresponding to $\hat{q}_1$, that is $p_1^d(\hat{q}_1) - p_1^s(\hat{q}_1) > 0$; hence there still is room for a weakly advantageous marginal trade between the two consumers, at any rate of exchange $\hat{p}_1 \in [p_1^s(\hat{q}_1), p_1^d(\hat{q}_1)]$, or even in general for a finite trade, under suitable restrictions on the allowable rates of exchange, depending on the amount already traded, the amount to be traded, and the graphs of the Marshallian demand and supply functions of commodity 1 for $q_1 > \hat{q}_1$. Given the quasi-linearity in commodity 2 of the utility functions, whatever the allowable rate of exchange between the two commodities at which any marginal (or allowable finite) trade occurs, the Marshallian demand and supply functions of commodity 1 are unaffected. Hence the Marshallian equilibrium price and quantity of commodity 1 are independent of the path followed by the exchange process; as a consequence, the total traded quantity of commodity 1 will always equal $q_1(p_1^M)$ when the exchange process eventually ceases, while the marginal rate of exchange at which the last marginal trade occurs will always be p_1^M. Hence, as Marshall correctly suggests, the rate of exchange p_1^M ought to be interpreted as the final rate to which the sequence of the rates at which the consumers have traded during the trading process necessarily converges, along a path which may exhibit no regularity other than the stated convergence; the total quantity of commodity 1 traded by the consumers, $q_1(p_1^M)$, ought instead to be interpreted as the quantity of commodity 1 to which the monotonically increasing sequence of the quantities cumulatively traded by the consumers during the exchange process necessarily converges. Finally, the total quantity of the money-like commodity 2 cumulatively traded by the consumers at the end of the trading process remains undetermined, its final value being however necessarily confined to the interval

$$\left[\int_0^{q_1(p_1^M)} p_{1i}^s(s_{1i})ds_{1i}, \int_0^{q_1(p_1^M)} p_{1j}^d(d_{1j})dd_{1j}\right],$$

where $i, j = 1,2$, i is s.t. $v_{1i}'(\omega_{1i}) = p_{1\min}^s$, while j is s.t. $v_{1j}'(\omega_{1j}) = p_{1\max}^d$.

Hence, in Marshall's model of an Edgeworth Box economy with a money-like commodity there is no counterpart of equation $(2'')$, appearing in Walras's model, where it provides the market-clearing condition for commodity 2; and, for the same reason, in Marshall's model there is nothing comparable to Walras' Law, even if, due to the bilateral character of any exchange, the total value of sales must always equal that of purchases for each consumer, hence for the whole economy.

3. Marshall's "temporary equilibrium" model

The formal analysis developed above supports the conclusions informally reached by Marshall in his Edgeworth Box artificial example with a money-like commodity ("nuts") and a commodity proper ("apples"). It is obvious, however, that this is just a provisional result for Marshall, whose aim evidently is to apply his method of analysis to a more realistic economy, with an arbitrary finite number of traders and commodities. Yet, while Marshall's objectives are indeed quite general, the analytical tools at his disposal remain quite limited: in fact, in developing his analysis of the so-called "temporary equilibrium of demand and supply" in Chapter II of Book V of the *Principles*, while explicitly referring to an exchange economy with any finite number of traders and commodities, Marshall puts forward (or, more precisely, informally illustrates) a model where he explicitly takes into account an arbitrary finite number of traders, but only two commodities at a time. As a consequence, the model illustrated in Chapter II of Book V, henceforth referred to as Marshall's "temporary equilibrium" model, can only represent a very partial generalization of the Edgeworth Box model of Appendix F, with the following features: the number of traders increases to $I > 2$; the number of commodities formally taken into consideration still remains $L = 2$; the money-like commodity becomes "money" proper, that is, the counterpart of any trade, or the "general purchasing medium" in the economy, whose marginal utility is assumed to be constant ([1961a], p. 335-336, 793); the other commodity is explicitly taken to be a consumers' good.

The last qualification requires some comment. In passing from the Edgeworth Box model with a money-like commodity to the "temporary equilibrium" model with money, Marshall further specifies the conditions under which the "constant marginal utility of money" assumption is empirically justified and substantially satisfied. In fact, to the already mentioned characteristic property of money of being in large supply and general use, Marshall now adds another condition, concerning however not money as such, but rather the commodity for which money is exchanged: "*The "constant marginal utility of money"] assumption is justifiable with regard to most of the market dealings with which we are practically concerned. When a person buys anything for his own consumption, he generally spends on it a small part of his total resources; while when he buys it for the purposes of trade, he looks to re-selling it, and therefore his potential resources are not diminished. In either case there is no appreciable change in his willingness to part with money. [...] The exceptions are rare and unimportant in markets for commodities [that is, consumers' goods]; but in markets for labour they are frequent and important. [...] The theory of buying and selling becomes therefore much more complex when we take account of the dependence of marginal utility on amount in case of money as well as of the commodity itself. The practical importance of this consideration is not very great*" ([1961a, p. 335-336).

In view of this passage, we can conclude that Marshall's "temporary equilibrium" model actually consists in a limited extension of his Edgeworth Box model with a money-like commodity to a pure-exchange, two-commodity economy with an arbitrary finite number of traders, that is, an economy $\mathcal{E}_{pe}^{2\times I,m} = \left\{\left(\Re_{+}^{2}, u_i(\cdot), \omega_i\right)_{i=1}^{I}\right\}$ with $I > 2$, where commodity 1 is a consumers' good, commodity 2 is money, and the marginal utility of commodity 2 is assumed to be constant. Even if, formally, the model can be said to apply to an entire pure-exchange economy with the specified characteristics, from a substantial point of view it actually describes the functioning of a single market, namely, the market where a given consumers' good is exchanged for money. This simply means that the model under discussion, though formally constructed as a general equilibrium model, actually provides the foundations of Marshall's partial equilibrium analysis of an isolated market. This ambiguity is not devoid of consequences.

Let us consider, in particular, the "constant marginal utility of money" assumption. Of the two conditions that, according to Marshall, justify this assumption, the first can be taken care of in the same way as before, by fixing a minimum endowment of money, m_i, for each $i = 1,...,I$. But the second cannot be formally accommodated into the model of an economy with only two commodities, one of which is money, for in such a model it is meaningless to suppose that each trader i's expenditure on the only consumers' good existing in the economy represents "a small part of his total resources". This is just an instance of the difficulties one necessarily encounters in trying to make formally precise Marshall's rich, but vague, empirical insights, while striving to keep the formal model as faithful as possible to Marshall's original presentation.

Similar remarks apply, in particular, to the idea of formalizing the behavior of those dealers or middlemen, supposedly buying with a view to re-selling, who are incidentally mentioned by Marshall in the passage quoted above: for the formal treatment of that sort of behavior, with its obvious strategic connotations, would require the use of a conceptual framework and an analytical apparatus which are entirely alien to Marshall's capabilities and interests. Hence, in the following, we shall rule out all strategic considerations, assuming instead that all traders engage in bilateral bargains, satisfying the following conditions: each bargain is regarded as a self-contained transaction by the two traders involved in it, so that each trader, in deciding whether to get engaged in a bargain, takes into account only the immediate effects of that bargain on his utility[29]. On top of this assumption, which is specific to the "temporary equilibrium" model, due to the existence in this model of a number of traders greater than two, we have to confirm here the same two assumptions already encountered in Marshall's Edgeworth Box model: precisely, in conformity with Marshall's verbal description of the exchange process, we shall assume that an individual bargain will only take place if it is weakly advantageous for the two traders involved in it, while no trader will stop trading as long as he can increase his utility by so doing.

Under the above assumptions, the generalization of the model of an Edgeworth Box economy with a money-like commodity, $\varepsilon_{EB}^{m} = \varepsilon_{pe}^{2\times 2,m}$, to the "temporary equilibrium" model of a pure-exchange economy with I consumers, $\varepsilon_{pe}^{2\times I,m}$, is immediate: in effect, all the analysis leading to equations (4') and (4") is independent of the number of traders in the economy, and consequently applies without change to the new context, except that now the number of traders in the economy is $I > 2$, instead of just 2 as before.

Yet, in spite of their formal similarity, it is nonetheless convenient to distinguish between the two cases: namely, when referring to the economy $\varepsilon_{pe}^{2\times I,m}$, rather than to the economy ε_{EB}^{m}, we shall rewrite equations (4') and (4") as

$$d_1^I(p_1^{I,M}) = s_1^I(p_1^{I,M}) \qquad (5')$$

or

$$d_1^I(p_1^{I,M}) - s_1^I(p_1^{I,M}) = 0, \qquad (5'')$$

it being understood that, in deriving equations (5') and (5"), the Marshallian aggregate demand and supply functions for commodity 1 are, respectively, $d_1^I(p_1^d) = \sum_{i=1}^{I} d_{1i}\left(p_{1i}^d\right)$, for $p_1^d = p_{1i}^d$, $i = 1,...,I$, and $p_1^d \in (0,\infty)$, and $s_1^I(p_1^s) = \sum_{i=1}^{I} s_{1i}\left(p_{1i}^s\right)$, for $p_1^s = p_{1i}^s$, $i = 1,...,I$, and $p_1^s \in [0,\infty)$, with $I > 2$ (rather than $I = 2$, as in the derivation of equations (6) and (7)). Further,

[29] Marshall's exclusion of all strategic considerations from his "temporary equilibrium" model is openly stressed by Berry in a private letter to Edgeworth, once again written at Marshall's suggestion. In trying to defend Marshall's stance from Edgeworth's criticism, Berry writes *inter alia*: “Your argument as to recontracts which would disturb temporary equilibrium, I found very interesting and it seems to me quite true, but I hardly think it bears directly on Marshall's chapter, where recontracts are tacitly excluded” ([1961b], p. 794).

$p_1^{I,M}$ is the Marshallian "temporary equilibrium" money price of commodity 1, while $q_1^I(p_1^{I,M}) = d_1^I(p_1^{I,M}) = s_1^I(p_1^{I,M})$ is the Marshallian "temporary equilibrium" quantity of commodity 1.

As we shall see, Marshall's final interpretation of equation (5') or (5") is essentially the same as that of equation (4') or (4"). Yet, Marshall's claims are not entirely justified: for, even if equations (5') and (5") are formally almost identical to equations (4') and (4"), their interpretation cannot be exactly the same as before. To clarify this point, let us first recall the essential features of Marshall's original presentation. Marshall, as it is customary for him, develops his "temporary equilibrium" model by means of an example. In this case, Marshall's illustration is taken "from a corn market in a country town", where "corn [...] of the same quality" is traded against "money", the former being measured in quarters and the latter in shillings (Marshall, 1961a, p. 332). Hence, in the light of Marshall's example, commodities 1 and 2 above should be interpreted as "corn" and "money", respectively, while the price of commodity 1 in terms of commodity 2 should be interpreted as the "money price of corn". In his illustration, Marshall summarizes the relevant aggregate "*facts*" concerning the corn market by means of the following "*table*" ([1961a], p. 333):

At the price	Holders will be willing to sell	Buyers will be willing to buy
37 *s*.	1000 quarters	600 quarters
36 *s*.	700 "	700 "
35 *s*.	600 "	900 "

From a discussion of these "facts", Marshall draws the following provisional conclusion: " *The price of 36 s. has thus some claim to be called the true equilibrium price: because if it were fixed on at the beginning, and adhered to throughout, it would exactly equate demand and supply (i.e. the amount which buyers were willing to purchase at that price would be just equal to that for which sellers were willing to take that price); and because every dealer who has a perfect knowledge of the circumstances of the market expects that price to be established. If he sees the price differing much from 36 s. he expects that a change will come before long, and by anticipating it he helps it to come quickly*" ([1961a], p. 333-4).

Here Marshall offers two different reasons for justifying the statement that "*the price of 36 s.*" is "*the true equilibrium price*". What is at first sight disconcerting is that neither argument is really consistent with Marshall's approach: for the first ambiguously oscillates between a Jevonsian and a Walrasian approach, while the second assumes an amount of knowledge on the part of some dealers that is wholly at variance with both Marshall's vision and theory.

The first argument has an explicitly conditional form: if some extreme form of "Jevons' Law of Indifference" were to hold, implying the constancy of the money price of commodity 1 over the whole trading process, assumed to be time-consuming, rather than simply across the different trades taking place at one and the same instant, then "the price of 36 *s*." would clear the market for that commodity. This argument might appear to suggest a distinctly Walrasian interpretation of both equation (5') (or (5")) and the price equilibrium concept implicit in it. But there is something unconvincing in this Walrasian reading of the price equilibrium concept: on the one hand, as we already know, Marshall does not believe in the truth of the premise of the proposed conditional statement, which sounds therefore as openly counterfactual in character[30]; on the

[30]As we have already seen in discussing Marshall's model of an Edgeworth Box economy, Marshall does not believe that a constant rate of exchange between the two commodities, representing the "true equilibrium rate", has any chance of prevailing over the trading process; in fact, referring to such "true position of equilibrium", he states that "there is no reason to suppose that it will be reached in practice" ([1961a], p. 791). As we shall see in a moment,

other hand, as can be seen by the second half of the sentence between parentheses, Marshall is far from accepting Walras's price-taking assumption, on which the Walrasian interpretation of the price equilibrium concept essentially rests[31].

Marshall's second argument is even more questionable: for if an inside dealer had a "perfect knowledge of the circumstances of the market", whatever the exact meaning of this expression, he would try to exploit such knowledge strategically, as Marshall himself seems to imply in the last sentence of the quoted passage. But then that dealer's behavior could not be the one predicted on the basis of Marshall's own simple non-strategic theory, as put forward in both Appendix F and Chapter II of Book V of the *Principles*, so that "the price of $36s$." could not be the equilibrium price, after all, and the alleged stabilizing effect of speculation would be far from proven, contrary to Marshall's implication[32].

Now, if Marshall's justifications of his own "temporary equilibrium" concept were really based on the above grounds, Marshall's efforts to build an original equilibrium model would be misplaced or self-defeating: in effect, if the proposed justification were the first, with its Walrasian flavor, Marshall's model should be discarded in favor of the much less cumbersome model put forward by Walras; if, instead, the proposed justification were the second, with its game-theoretic flavor, Marshall's model should be discarded since it would be wholly unable to cope with the issues at stake.

But really it is not Marshall's intention to support his "temporary equilibrium" notion by means of either one of the arguments tentatively put forward in the quoted passage: in fact, in the immediately following sentence, Marshall himself takes care to disavow them both. As to the second, based on the supposition that some dealers may possess a "perfect knowledge" of the market conditions, he writes: "*It is not indeed necessary for our argument that any dealers should have a thorough knowledge of the circumstances of the market*"([1961a], p. 334).

As to the first, based on the joint use of one extreme version of "Jevons' Law of Indifference" and the market-clearing condition, Marshall explains that, precisely because the dealers, far from being perfectly informed, actually have a very limited, or even grossly mistaken, knowledge of the circumstances of the market, a number of bilateral bargains will be struck at prices different from the equilibrium one. Yet, according to Marshall, in spite of all such trades occurring at non-equilibrium prices, the market will tend to close on a price not far from the equilibrium price ($36s$.), while the total amount of corn traded will eventually approximate the equilibrium quantity (700 quarters). Specifically, Marshall writes: "*Many of the buyers may perhaps underrate the willingness of the sellers to sell, with the effect that for some time the price rules at the highest level at which any buyers can be found; and thus* 500 *quarters may be sold before the price sinks below* $37s$. *But afterwards the price must begin to fall and the result will still*

Marshall is similarly convinced that there is no reason why, in his "temporary equilibrium" model, the money price of the consumers' good concerned should be supposed to remain constant over the trading process analyzed therein. Indeed, Marshall appears sometimes to believe that the standard (i.e., instantaneous) version of "Jevons' Law of Indifference" holds approximately true in "perfect markets": "Thus the more nearly perfect a market is, the stronger is the tendency for the same price to be paid for the same thing *at the same time* in all parts of the market [...]" ([1961a], p. 325; italics added). But this has nothing to do with assuming the constancy of price *over time*.

[31]While the first half of the bracketed expression (that is, "the amount which buyers were willing to purchase at that price") may sound Walrasian, since the "buyers" may be viewed as price-takers and quantity-adaptors, the second half (that is, "would be just equal to that for which sellers were willing to take that price") cannot for sure be so interpreted, since here the "sellers" are supposed to decide whether or not to accept a certain price, given a certain quantity of output, which is surely not a competitive behavior in the Walrasian sense.

[32]As clearly emerges from the starting paragraph of Chapter 3 of Book V of the *Principles*, which immediately follows the Chapter devoted to the "temporary equilibrium" model, Marshall is perfectly aware that "[e]ven in the corn-exchange of a country town on a market-day the equilibrium price is affected by calculations of the future relations of production and consumption", hence by expectations and speculation ([1961a], p. 337). But all these aspects, however important in the real world, are deliberately left out of the formal model of "temporary equilibrium".

probably be that 200 more quarters will be sold, and the market will close on a price of about 36s. For when 700 quarters have been sold, no seller will be anxious to dispose of any more except at a higher price than 36s., and no buyer will be anxious to purchase any more except at a lower price than 36s. In the same way if the sellers had underrated the willingness of the buyers to pay a high price, some of them might begin to sell at the lowest price they would take, rather than have their corn left on their hands, and in this case much corn might be sold at a price of $35s$.; but the market would probably close on a price of 36s. and a total sale of 700 quarters ([1961a], p. 334).

We have here a distinctly non-Walrasian equilibration process, since out-of-equilibrium trades are explicitly allowed for, though not formally modelled. And yet the process is said to converge to a well-determined price of corn in terms of money and a well-determined total traded quantity of corn, where such price and quantity incidentally coincide with the Walrasian equilibrium ones. Once again, as he had already done in the context of the Edgeworth Box economy, Marshall explains that also in this case the determinateness of equilibrium crucially depends on the "constant marginal utility of money" assumption: "*In this illustration there is a latent assumption which is in accordance with the actual conditions of most markets; but which ought to be distinctly recognized in order to prevent its creeping into those cases in which it is not justifiable. We tacitly assumed that the sum which purchasers were willing to pay, and which sellers were willing to take, for the seven hundredth quarter would not be affected by the question whether the earlier bargains had been made at a high or low rate*" ([1961a], p. 334).

But is Marshall justified in supposing that the "constant marginal utility of money" assumption is sufficient for granting equilibrium determinateness in a pure-exchange economy with many traders, $\varepsilon_{pe}^{2\times I,m}$ with $I > 2$, as it was in an Edgeworth Box economy with a money-like commodity, ε_{EB}^{m}? The answer is: not quite. In fact, in the model of an Edgeworth Box economy with a money-like commodity, the sharp result which has been obtained concerning p_1^M, the Marshallian equilibrium price of commodity 1 in terms of commodity 2, crucially depends on the existence of only two traders in the economy: for in that case the marginal rate of exchange at which the last marginal trade occurs necessarily coincides with both the marginal demand price of the only marginal buyer, $p_1^d\left(q_1(p_1^M)\right)$, and the marginal supply price of the only marginal seller, $p_1^s\left(q_1(p_1^M)\right)$; hence it also necessarily coincides with p_1^M, which can therefore be legitimately interpreted as the final rate to which the sequence of the rates at which the traders have traded during the exchange process necessarily converges.

But in Marshall's "temporary equilibrium" model there are more than two traders in the economy; hence, in general, not only there might exist more than one marginal buyer or seller, but there might also be some sellers that are not marginal, in the sense that the minimum supply prices at which their Marshallian direct supply functions become perfectly price-inelastic are less than $p_1^{I,M}$. Under such circumstances, however, we can no longer be sure the marginal price at which the last marginal trade occurs necessarily coincides with $p_1^{I,M}$: whether or not this holds true depends on the path followed by the exchange process, specifically on the order of the matchings between pairs of traders, that is, on something on which Marshall's theory has nothing to say. In Marshall's "temporary equilibrium" model, therefore, while the total quantity of commodity 1 traded in the market will still certainly converge to the Marshallian "temporary equilibrium" quantity, $q_1(p_1^M)$, it is no longer true that the sequence of the money prices of commodity 1 at which the traders buy and sell that commodity during the trading process necessarily converges to the Marshallian "temporary equilibrium" price, $p_1^{I,M}$.

4. Marshall's pure-exchange models: limitations and extensions

While Marshall's model of the Edgeworth Box economy is obviously propedeutical to his "temporary equilibrium" model, the latter is in turn propedeutical to his normal equilibrium models, which absorb by far the largest part of Marshall's attention in Book V of the *Principles* and can rightly be regarded as the crowning of the Marshallian theory of value. Yet it would be wrong to underrate the role of Marshall's pure-exchange models, for they provide the foundations upon which the whole of Marshall's price theory is built, fixing at the same time the boundaries within which it can hope to expand. As a matter of fact, unlike many of his followers and interpreters, Marshall himself is well aware of the fundamental role played by his pure-exchange models in the overall structure of his thought, even if he is apparently willing to acknowledge it in private correspondence only. For instance, in reproaching Edgeworth (1891a) for wrongly bringing a charge of indeterminacy against his model of barter with a money-like commodity, Marshall ([961b], p. 797) does not hesitate to assert that, if the "error" mistakenly pointed out by Edgeworth were in effect true, it "would justly shake the credit of a very great part of his [i.e., Marshall's] book". In his pure-exchange, two-commodity models Marshall wants to show how an equilibrium comes to be established as the final outcome of a realistic process of exchange in "real" time, where trades can actually take place at out-of-equilibrium rates of exchange or prices. This program inevitably raises the issue of equilibrium determinacy. Marshall's solution, as we have seen, consists in imposing some related restrictions on both the traders' utility functions, which are assumed to be quasi-linear in one of the two commodities, and the nature of the commodities themselves, one of which is interpreted as a money-like commodity or money *tout court*. By so proceeding, Marshall solves the equilibrium determinacy problem in the Edgeworth Box model with a money-like commodity, in the special sense specified in subsection 4.2, and almost solves it in the "temporary equilibrium" model, as explained in subsection 4.3.

But, at the same time, Marshall inexorably restrains the scope of his analysis: for his suggested solution of the equilibrium indeterminacy problem only applies when no more than one commodity proper is explicitly accounted for in the model, so that the only unknowns to be determined boil down to the money price and the quantity traded of that single commodity proper (as we have seen, not even the quantity of money traded in equilibrium can be determined in Marshall's model). As a matter of fact, Marshall's approach can be formally extended to a multi-market pure-exchange economy, where an arbitrary finite number of commodities are traded for money. Yet, in such a generalized context, "Marshall's fundamental property", on which Marshall's results in his pure-exchange models with only one commodity proper crucially depend, can only be preserved if one is willing to assume that the traders' utility functions are not only quasi-linear in money, but also additively separable in all their arguments, i.e., all commodities proper and money; this means, however, that the multi-market economy actually turns out to be made up of a number of separate markets, lacking any essential interrelation and behaving as if they were isolated form each other[33].

We can conclude, therefore, that there is no way to extend to a multi-commodity world, made up of many interrelated markets, the results achieved by Marshall within his one-commodity world, consisting in the isolated market where the only commodity proper explicitly

[33]Marshall is well aware of the fundamental role played by the additive separability of the traders' utility functions in his pure-exchange models. Yet, in his typical style, instead of openly acknowledging the irreplaceable *analytical* role of that assumption in his theoretical construction, he prefers to justify it on *empirical* grounds. In fact, in Note XII *bis* of the Mathematical Appendix of the *Principles,* where Marshall discusses his model of an Edgeworth Box economy with a commodity proper ("apples") and a money-like commodity ("nuts"), defending it from Edgeworth's criticism, he writes: "*Prof. Edgeworth's plan of representing U and V [the traders' utility functions] as general functions of x and y [the quantities of the two commodities] has great attraction to the mathematician; but it seems less adapted to express the every-day facts of economic life than that of regarding, as Jevons did, the marginal utilities of apples as functions of x simply*" ([1961a], p. 845).

contemplated by the model is traded for money: Marshall's analysis remains necessarily confined to the partial equilibrium framework dictated by his explanatory aims and consequent choice of assumptions, while Walras's general equilibrium analysis stands well beyond reach. Of course, production phenomena can be brought into the picture: this is precisely what Marshall does by developing his normal equilibrium models, where production of a consumers' good plays a fundamental role in explaining the functioning of the supply side of the market. But also in this case the partial equilibrium framework cannot be overcome.

Conclusions

In this paper we have squarely faced the long-standing issue of the foundations of modern price theory, specifically contrasting the received view according to which Walras's and Marshall's approaches to price theory, while differing in scope, are basically similar in their aims, presuppositions, and results. By focusing on a special kind of economy (the pure-exchange, two-commodity economy), which has been formally studied by both Walras and Marshall with the help of similar tools, we have been able to precisely identify the differences between the two approaches. First, the two economists have been shown to widely differ from one another in the basic assumptions on which they ground their respective investigations of the trading process: as a matter of fact, it turns out that Walras's very conception of a competitive economy is largely at variance with Marshall's. Secondly, it has been shown that, starting from such different sets of assumptions, the two authors arrive at entirely different models of the pure-exchange, two-commodity economy. Precisely, by reducing the trading process to a purely virtual process in "logical" time, Walras arrives at a well-defined notion of "instantaneous" equilibrium, which can be easily extended to more general contexts (such as pure-exchange and production multi-commodity economies). On the contrary, by making a few further assumptions on the characteristics of the traders and the nature of the commodities involved, one of which must be money or a money-like commodity, Marshall can indeed show that a determinate (or almost determinate) equilibrium emerges from a process of exchange in "real" time with observable out-of-equilibrium trades; but his analysis cannot be significantly generalized beyond the partial equilibrium framework in which it is necessarily couched from the beginning.

Hence, to conclude, our comparison between Walras's and Marshall's approaches to price theory seems to confirm that, given the requirement of equilibrium determinacy, there indeed exists a trade-off between realism and scope of the analysis: for Marshall can buy a more realistic interpretation of both the equilibration process and the equilibrium construct than Walras's only at the cost of giving up Walras's pretence to develop a truly general analysis of a system of interrelated markets.

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